# Economic Foundations and Applications of Risk

## Part A. Foundations

### Chapter 1: Expected-Utility Theory

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# Syllabus

- 1.1 Introduction
- 1.2 The axioms
- 1.3 The vNM theorem
- 1.4 Basic properties of vNM utility
- 1.5 Risk preferences
- 1.6 Indifference curves of vNM utility functions

# 1.1 Introduction

$EU = p \cdot u(x_1) + (1-p) \cdot u(x_2)$

- In their seminal work “Theory of Games and Economic Behavior” (1944), **John von Neumann** and **Oscar Morgenstern** develop the axiomatic foundations of Expected-Utility Theory.[1]
- We will first study their **axioms** (1.2)...
- ...from which we will derive the pivotal **vNM theorem** (1.3).
- Then we will look at some **basic properties** (1.4) of vNM utility functions ...
- ...and introduce the concept of **risk preferences** (1.5).
- We will close by looking at the **indifference curves** of vNM utility functions in the so-called 2-states-of-the-world diagram (1.6).

[1]Von Neumann, J. and Morgenstern, O. (1944); Theory of Games and Economic Behavior; Princeton, N.J.; Princeton University Press

# 1.2 The axioms

## Some definitions

- Let $\mathbf{L}$ be a set of lotteries $\{\mathbf{L}_1, ..., \mathbf{L}_n\} \equiv \mathbf{L}$.
- Let there be a **"standard lottery"** $(1-u, u; x_{min}, x_{max})$,
  - where $x_{min}$ and $x_{max}$ are chosen such that the following holds:

$$x_{min} \leq x \;\; \forall \, x \in \mathbf{X}; \quad x_{max} \geq x \;\; \forall \, x \in \mathbf{X},$$

  - where $\mathbf{X}$ is the matrix consisting of the payout vectors $\mathbf{X}_i$ pertaining to lotteries $\mathbf{L}_i \in \mathbf{L}$,
  - and where $u = Prob(x_{max})$.

## Axiom 1: Ordering of lotteries

- This axiom is sometimes referred to as the **“rationality axiom”.** It is perfectly analogous to similar axioms in standard micro theory under certainty.
- **Completeness**
  - $\forall\,(\mathbf{L}_i,\mathbf{L}_j)\in(\mathbf{L}\times\mathbf{L}):\quad \mathbf{L}_i\succeq\mathbf{L}_j\vee\mathbf{L}_j\succeq\mathbf{L}_i$
  - For any two given choices, an individual will always be able to tell which one she likes better or whether she is indifferent.
- **Transitivity**
  - $\forall\,(\mathbf{L}_i,\mathbf{L}_j,\mathbf{L}_k)\in(\mathbf{L}\times\mathbf{L}\times\mathbf{L}):\quad(\mathbf{L}_i\succeq\mathbf{L}_j\wedge\mathbf{L}_j\succeq\mathbf{L}_k)\Rightarrow\mathbf{L}_i\succeq\mathbf{L}_k$
  - If an individual likes oranges better than apples and apples better than pears, we can infer that she likes oranges better than pears.
- **Reflexivity**
  - $\forall\,\mathbf{L}_i\in\mathbf{L}:\quad\mathbf{L}_i\succeq\mathbf{L}_i$
  - 1 lb of apples is no worse than 1 lb of (the same) apples.

## Axiom 2: Preferences over probabilities

- Let there be standard lotteries $\mathbf{L}_i = (1 - u_i, u_i; x_{min}, x_{max}) \in \mathbf{L}$
- Then: $\mathbf{L}_1 \succeq \mathbf{L}_2 \Leftrightarrow u_1 \geq u_2$.
- This axiom is akin to the axiom of **local non-satiation,** which we know from standard consumer theory.
- It says that, given a choice between two standard lotteries, individuals will prefer the one with more probability mass on $x_{max}$.

$$\frac{du}{dx} > 0$$

## Axiom 3: Continuity

- $\forall x \in [x_{min}; x_{max}] : \quad \exists\, u(x) \in [0; 1]$ such that

$$x \sim (1 - u(x), u(x); x_{min}, x_{max}).$$

- This says that for any given (certain) payout, it is always possible to construct a standard lottery such that an individual is **indifferent** between the two.
- Example:
  - $x_{min} = 0$, $x_{max} = 10.000$, $x = 1.000$
  - In this case, the individual is indifferent between getting a certain payment of 1.000 or getting 10.000 with probability u(1.000).

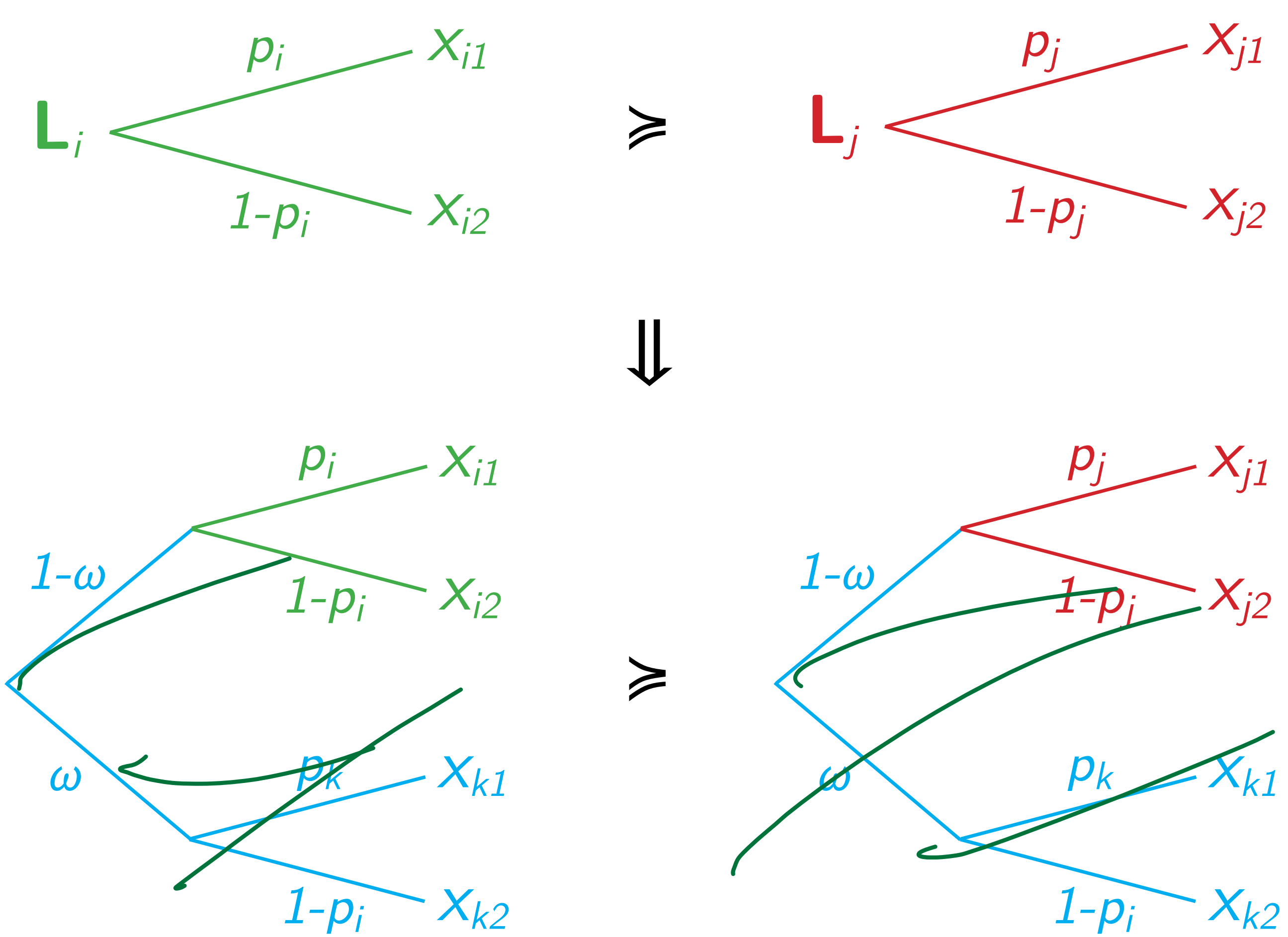
# Axiom 4: Independence

- $\forall\ (\mathbf{L}_i, \mathbf{L}_j, \mathbf{L}_k) \in (\mathbf{L} \times \mathbf{L} \times \mathbf{L})$ with $\mathbf{L}_i \succeq \mathbf{L}_j$ and $\forall\ \omega \in [0;1]$:

$$(1-\omega, \omega; \mathbf{L}_i, \mathbf{L}_k) \succeq (1-\omega, \omega; \mathbf{L}_j, \mathbf{L}_k)$$

- This looks **rather plausible.**
  - With both lotteries the individual will get $\mathbf{L}_k$ with probability $\omega$.
  - With the first lottery, she will get $\mathbf{L}_i$ with probability $(1-\omega)$
  - With the second lottery, she will only get $\mathbf{L}_j$ (which, by assumption, is equal or worse than $\mathbf{L}_i$) with the same probability $(1-\omega)$.
  - Hence, the second lottery should not be preferred.
- **Empirical findings** suggest, however, that this independence axiom may in some instances be **problematic**.
- Indeed, the axiom presupposes that:
  - Individuals can handle compound lotteries (lotteries over lotteries).
  - Individuals are aware that there are no complement effects between lotteries.

- Consider this event-tree figure:



# 1.3 The vNM theorem

$Eu = p \cdot u(x_1) + (1-p) \cdot u(x_2)$

## Definition 1.1: vNM utility function

- A **vNM utility function** is a function $U(\mathbf{L}_i)$ such that

$$U(\mathbf{L}_i) = \sum_j p_{ij} u(x_{ij}) = \mathrm{E}(u(\mathbf{x}_i)) \equiv \mathrm{E}u(\mathbf{x}_i),$$

- where $\mathbf{L}_i \in \mathbf{L}$, $p_{ij}$ is the probability of payout $x_{ij} \in \mathbf{x}_i$, and $u(\mathbf{x}_i)$ is given by axiom 3.

**Comments:**

- Note that $u(\mathbf{x}_i)$ is a probability function (see axiom 3)...
- ...but can also be interpreted as a **"Bernoulli utility function"**.
  - Why does this make sense?
- A vNM utility function is the **expected value of an individual's utility** when facing lottery $\mathbf{L}_i$.

## Theorem 1.1: vNM theorem

Any vNM-rational individual (i.e. satisfying axioms 1–4) will be acting **as if she was maximizing a vNM utility function,** when choosing between lotteries:

$$\mathbf{L}_i \succeq \mathbf{L}_j \Leftrightarrow U(\mathbf{L}_i) \geq U(\mathbf{L}_j) \Leftrightarrow$$

$$\mathbf{L}_i^* = \text{argmax}\, U(\mathbf{L})$$

**Comments:**

- This means that when **choosing the optimal lottery,** an individual will maximize the expected value of her utility.
- Note that the optimal $\mathbf{L}_i^*$ automatically determines the **optimal action** $a_i^*$ (see 0.Introduction, slide 14).

# Proof: The vNM theorem

- **WOLOG,** we will provide a proof for the simplest case: A lottery $\mathbf{L} = (1-p, p; x_1, x_2)$ with only two possible outcomes, $x_1$ and $x_2$.
- **Proof idea:** Show that for any lottery $\mathbf{L}$ there exists a probability, $U(\mathbf{L}) = (1-p) \cdot u(x_1) + p \cdot u(x_2)$, such that

$$\mathbf{L} \sim (1 - U(\mathbf{L}), U(\mathbf{L}); x_{min}, x_{max}).$$

- **Proof:**
  - Axiom 3: $x_1 \sim (1 - u(x_1), u(x_1); x_{min}, x_{max}) \equiv \mathbf{I}(x_1)$
  - Axiom 3: $x_2 \sim (1 - u(x_2), u(x_2); x_{min}, x_{max}) \equiv \mathbf{I}(x_2)$
  - Axiom 4: $\mathbf{L} \sim (1-p, p; \mathbf{I}(x_1), x_2)$
  - Axiom 4: $\mathbf{L} \sim (1-p, p; \mathbf{I}(x_1), \mathbf{I}(x_2))$
  - Plugging in $\mathbf{I}(x_1)$ and $\mathbf{I}(x_2)$:
    $\mathbf{L} \sim (1-p, p; [(1-u(x_1), u(x_1); x_{min}, x_{max})], [(1-u(x_2), u(x_2); x_{min}, x_{max}))]$

- **Proof (continued):**
  - Add up the probabilities for $x_{max}$ and $x_{min}$:
    - $\text{Prob}(x_{max}) = (1-p) \cdot u(x_1) + p \cdot u(x_2)$
    - $\text{Prob}(x_{min}) = (1-p) \cdot (1-u(x_1)) + p \cdot (1-u(x_2))$
      $= 1 - [(1-p) \cdot u(x_1) + p \cdot u(x_2)]$
      $= 1 - \text{Prob}(x_{max})$
  - Define: $\text{Prob}(x_{max}) = U(\mathbf{L})$ and $\text{Prob}(x_{min}) = 1 - U(\mathbf{L})$
  - Hence: $\mathbf{L} \sim (1-U(\mathbf{L}), U(\mathbf{L}); x_{min}, x_{max})$
    - With $U(\mathbf{L}) = (1-p) \cdot u(x_1) + p \cdot u(x_2)$.
  - QED.

# 1.4 Basic properties of vNM utility

## Transformations

- A **Bernoulli utility function** $u(\mathbf{x}_i)$ is unique up to a positive *linear* transformation.
  - If $u$ and $v$ are Bernoulli utility functions that represent the same preferences ...
  - ...then there exist constants $a$, $b$, with $a \in \mathbb{R}$ and $b \in \mathbb{R}_+$ ...
  - ...such that $v(\mathbf{x}_i) = a + bu(\mathbf{x}_i)$.
- A **vNM utility function** $U(\mathbf{L}_i)$ is unique up to a positive *monotonic* transformation.
  - More general than positive linear transformations.
  - Same assumption as for utility functions in standard consumer theory.
  - For example: $U(\mathbf{L}_i) = \sum_j p_{ij} u(x_{ij})$ and $V(\mathbf{L}_i) = \exp[\sum_j p_{ij} u(x_{ij})]$ represent the same preferences.

# 1.5 Risk preferences

## Definitions of concave functions

**Definition 1.2: Concave functions**

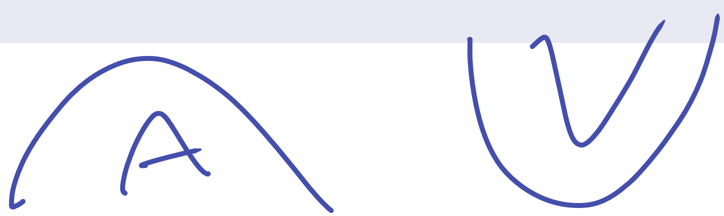
1. A function $f : \mathbb{R}^N \to \mathbb{R}$ is **(strictly) concave** if $\forall (x_1, x_2) \in \mathbb{R}^N$ and $\forall k \in [0; 1]$ :

$$f[kx_1 + (1-k)x_2] \geq (>)\ kf(x_1) + (1-k)f(x_2).$$

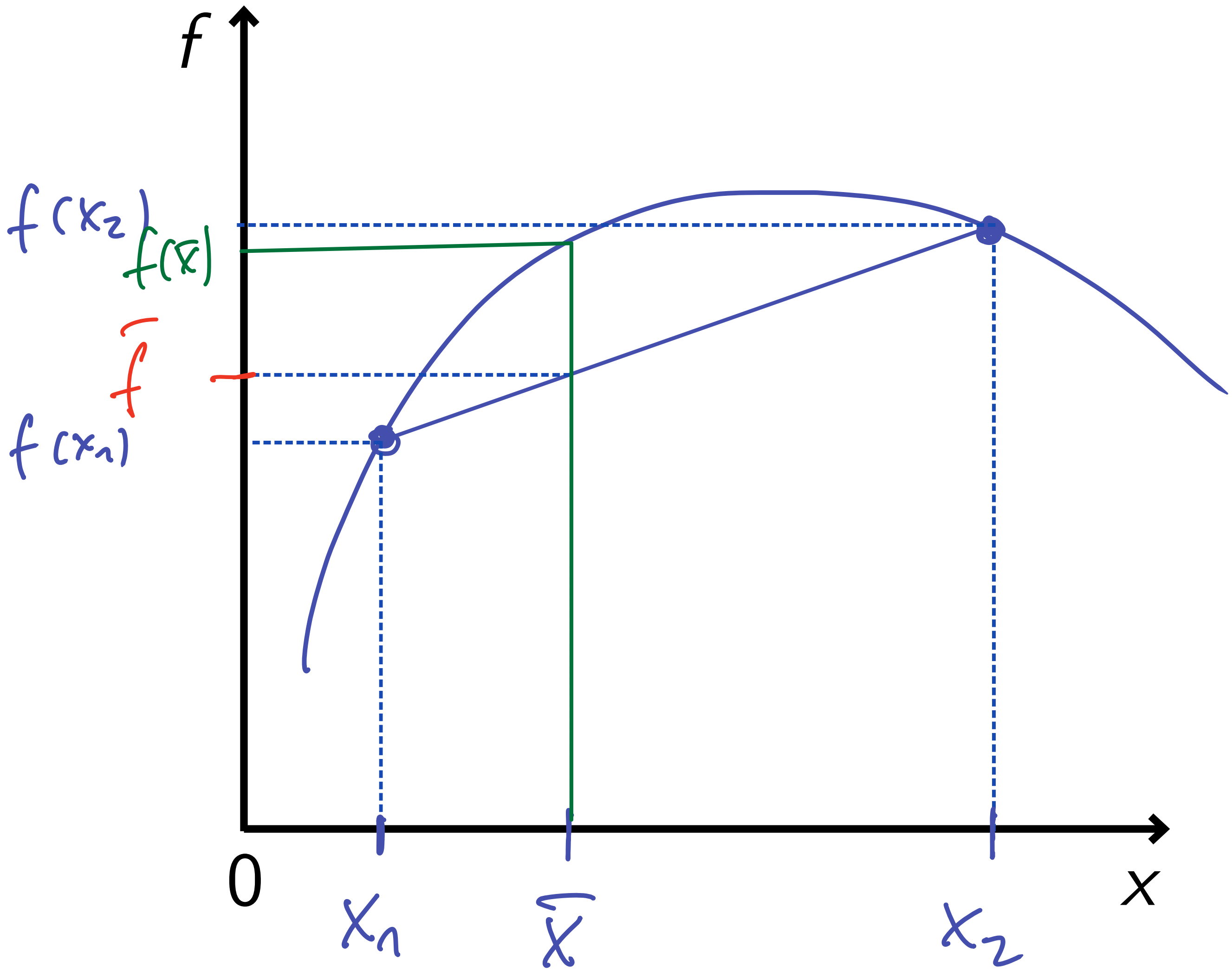
2. For at least three times continuously differentiable functions $f$, $f$ is **strictly concave** if

$$f''(x) < 0 \quad \forall\, x \in \mathbb{R}^N.$$

Mnemonic



# A concave function



## Definitions of convex functions

### Definition 1.3: Convex functions

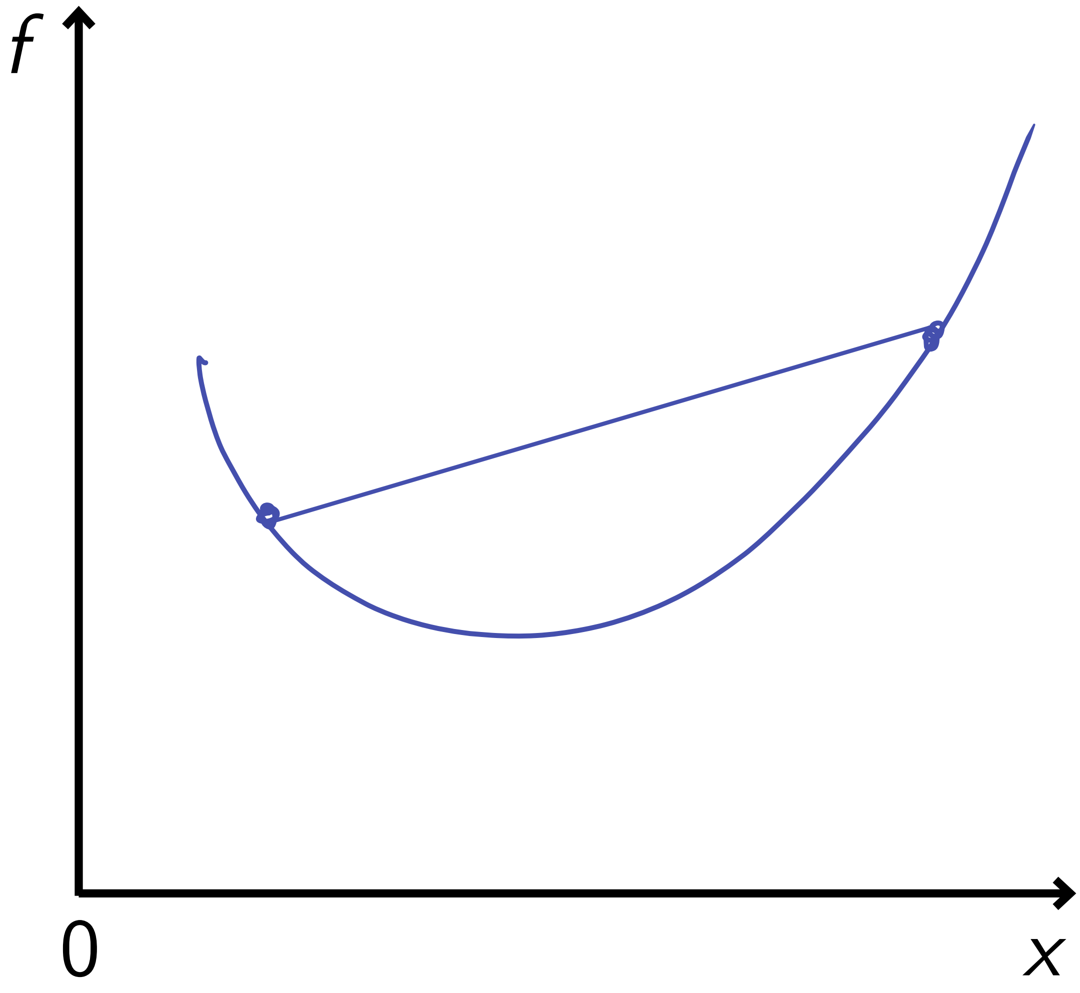
1. A function $f : \mathbb{R}^N \to \mathbb{R}$ is **(strictly) convex** if $\forall\,(x_1, x_2) \in \mathbb{R}^N$ and $\forall k \in [0;1]$ :

$$f[kx_1 + (1-k)x_2] \leq (<)\ kf(x_1) + (1-k)f(x_2).$$

2. For at least three times continuously differentiable functions $f$, $f$ is **strictly concave** if

$$f''(x) > 0 \quad \forall\, x \in \mathbb{R}^N.$$

# A convex function



## Definitions of linear functions

### Definition 1.4: Linear functions

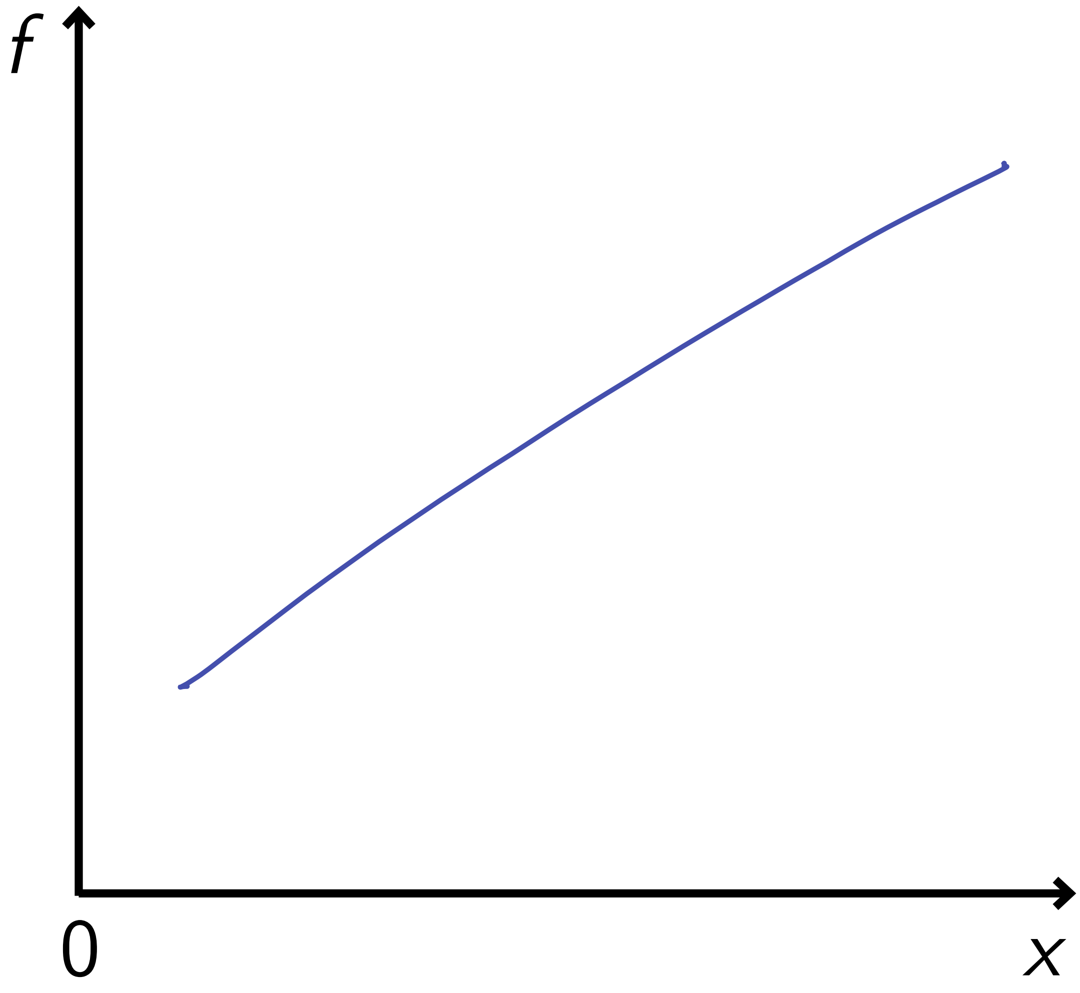
1. A function $f : \mathbb{R}^N \to \mathbb{R}$ is **linear** if $\forall (x_1, x_2) \in \mathbb{R}^N$ and $\forall k \in [0; 1]$ :

$$f[kx_1 + (1-k)x_2] = kf(x_1) + (1-k)f(x_2).$$

2. For at least twice continuously differentiable functions $f$, $f$ is **linear** if

$$f''(x) = 0 \quad \forall\, x \in \mathbb{R}^N.$$

# A linear function



## Definition of risk preferences

### Definition 1.5: Risk aversion

An individual with utility function $u$ is said to be **risk-averse** if she prefers the expected value of a lottery $\mathbf{L}$ over the lottery itself:

$$\mathrm{E}[u(\mathbf{L})] < u[\mathrm{E}(\mathbf{L})]$$

### Definition 1.6: Risk love

An individual with utility function $u$ is said to be **risk-loving** if she prefers a lottery $\mathbf{L}$ over its expected value:

$$\mathrm{E}[u(\mathbf{L})] > u[\mathrm{E}(\mathbf{L})]$$

## Definition 1.7: Risk neutrality

An individual with utility function $u$ is said to be **risk-neutral** if she is indifferent between a lottery $\mathbf{L}$ and its expected value:

$$\mathrm{E}[u(\mathbf{L})] = u[\mathrm{E}(\mathbf{L})]$$

## Risk preferences and the shape of the utility function

### Theorem 1.2: Concave utility functions imply risk aversion

A vNM-rational individual with **increasing** and **concave utility function** $u$ is **risk-averse**. ↳ non-satiation ↳ risk aversion

$$u'(\mathbf{x}) > 0 \ \wedge \ u''(\mathbf{x}) < 0 \quad \Longleftrightarrow \quad \mathrm{E}[u(\mathbf{L})] < u[\mathrm{E}(\mathbf{L})]$$

**Comments:**

- It is **typically assumed** that (human) individuals have concave utility functions, i.e. that they are **risk-averse.**
- For other entities (such as firms, organizations, or governments) this assumption is often relaxed.
- The assumption of increasing utility in $\mathbf{x}$ assures the basic rationality principle of non-satiability.

### Theorem 1.3: Convex utility functions imply risk love

A vNM-rational individual with **increasing and convex utility function** $u$ is **risk-loving**.

$$u'(\mathbf{x}) > 0 \wedge u''(\mathbf{x}) > 0 \iff \mathrm{E}[u(\mathbf{L})] > u[\mathrm{E}(\mathbf{L})]$$

### Theorem 1.4: Linear utility functions imply risk neutrality

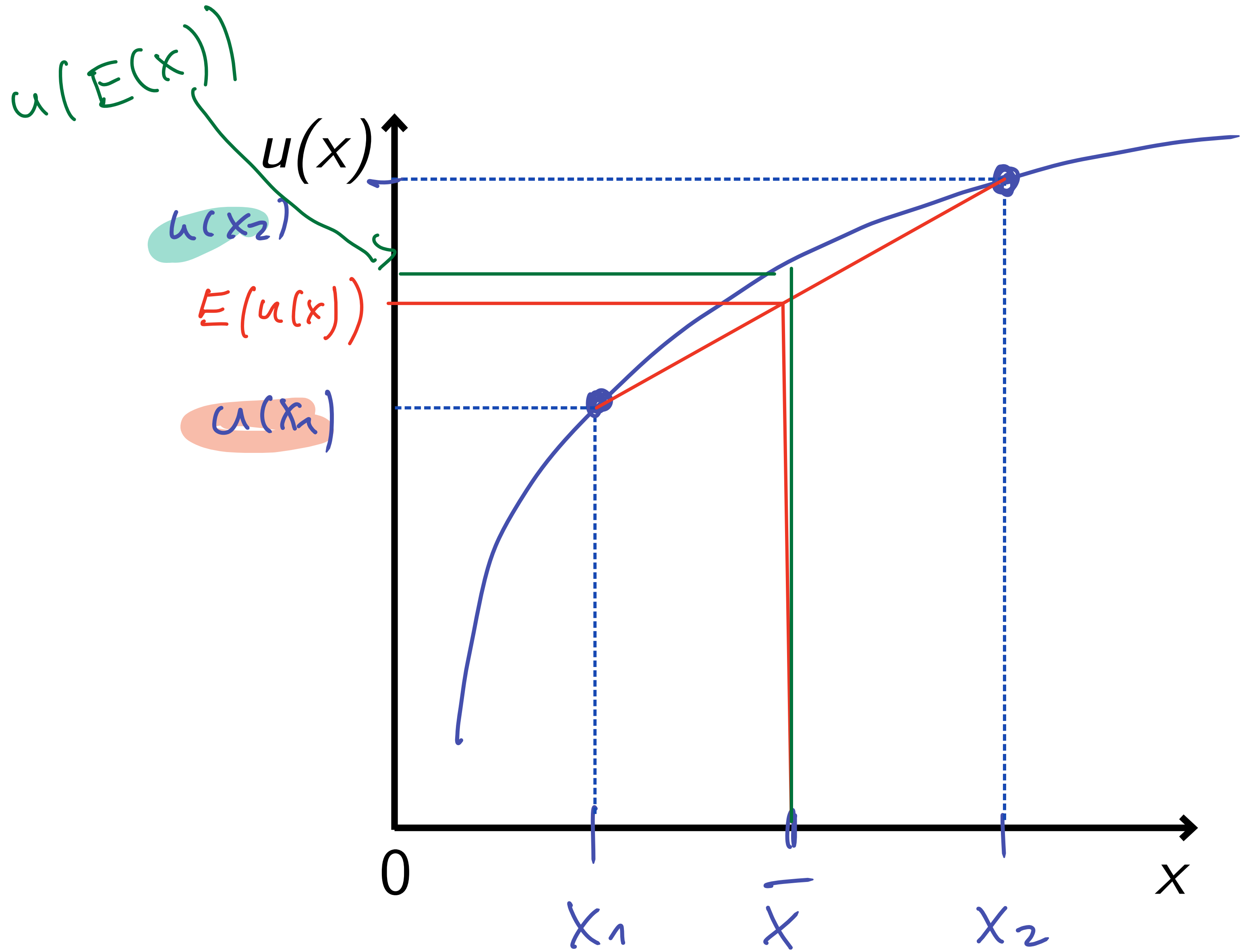
A vNM-rational individual with **increasing and linear utility function** $u$ is **risk-neutral**.

$$u'(\mathbf{x}) > 0 \wedge u''(\mathbf{x}) = 0 \iff \mathrm{E}[u(\mathbf{L})] = u[\mathrm{E}(\mathbf{L})]$$

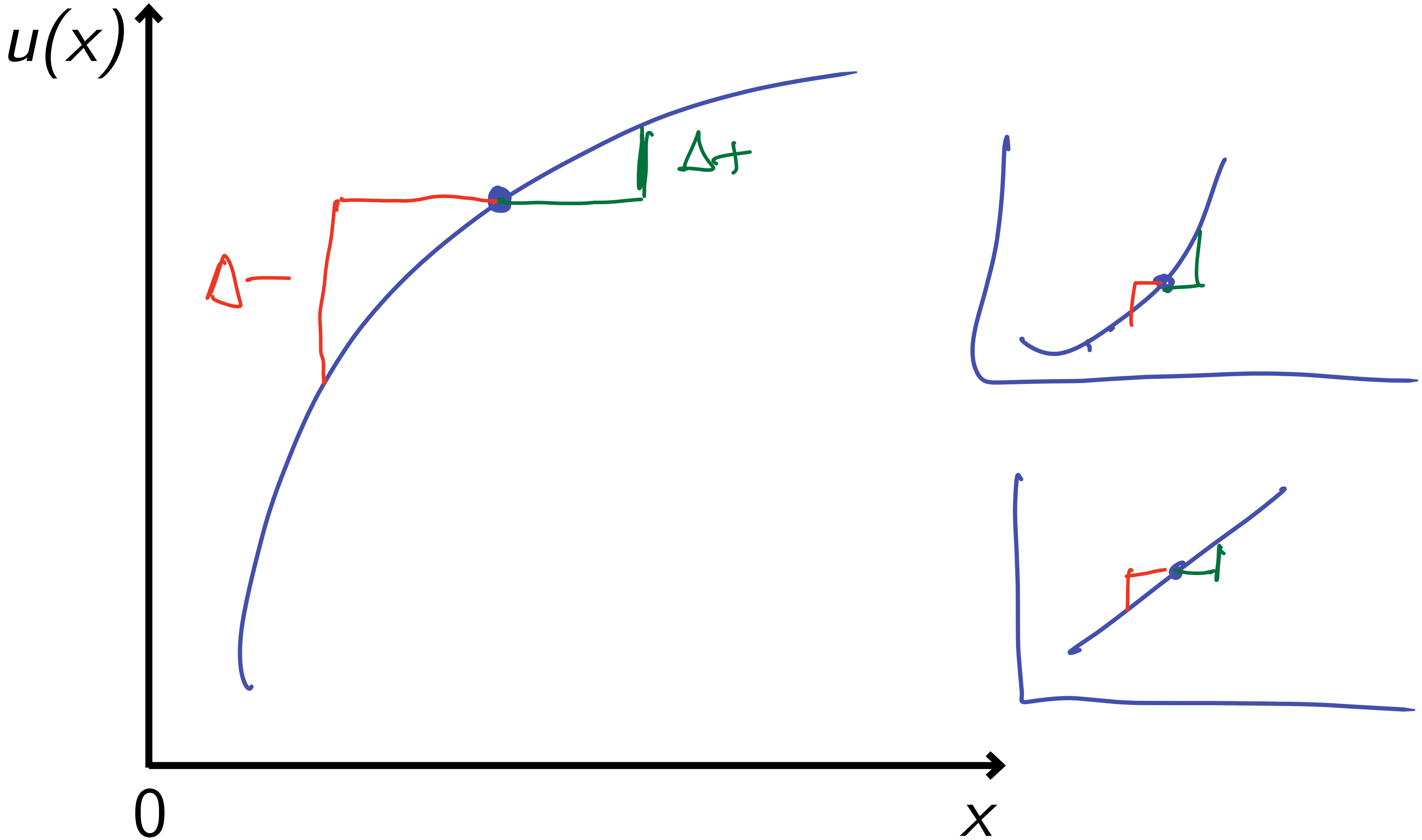
## Proof: Jensen's Inequality

- **WOLOG,** we will concentrate on the proof for **concave utility functions** (the standard case).
- The proofs for convex and linear utility functions are perfectly analogous.
- **Proof idea:** One can show that for any concave function $u(\mathbf{x})$ the following holds: $\mathrm{E}[u(\mathbf{x})] \leq u[\mathrm{E}(\mathbf{x})]$.
- You will prove Jensen's Inequality by means of a Taylor approximation in one of the next tutorials.
- Today, we will just tackle the (far more intuitive) **graphical "proof".**

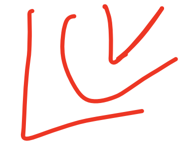
## Graph: Jensen's Inequality I



# Graph: Jensen's Inequality II



# 1.6 Indifference curves of vNM utility functions



- The indifference curves of vNM utility functions follow the same logic as that of standard utility functions.
- In the very simple case of two possible outcomes with $\mathbf{L} = (1-p, p; x_1, x_2)$, the indifference curves can be depicted in a so-called **"2-states-of-the-world" diagram.**
- The slope of the indifference curve equals the **marginal rate of substitution (MRS)**
  - The MRS indicates the rate at which an individual is willing to exchange income in state 2 for income in state 1.
  - $U(x_1, x_2) = (1-p)u(x_1) + pu(x_2) \iff$
  - $dU = (1-p)u'(x_1)dx_1 + pu'(x_2)dx_2 = 0 \iff$
  - $\text{MRS} \equiv \frac{dx_2}{dx_1} = -\frac{(1-p)u'(x_1)}{pu'(x_2)}$
- For **risk-averse individuals,** indifference curves are **convex.**
  - What about risk-loving and risk-neutral individuals?

## Graph: 2-states-of-the-world diagram