Economic Foundations and Applications of Risk

Part A. Foundations

Chapter 2: Measures of Risk Aversion

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Syllabus

2.1 Introduction

- In this chapter we will examine **how to measure risk aversion**. . .
- . . . and how to compare individuals with respect to the degree of their aversion to risk.
- We will first define the important concepts of **certainty equivalence** and the **risk premium** (2.2).
- Then we will define **local measures** of risk aversion (Pratt, 1960; and Arrow, 1970) (2.3) . . .
- ...and finish this chapter with **global measures** of risk aversion (Pratt, 1964) (2.4).
- We limit our attention to lotteries that generate monetary payouts (or, equivalently, that generate outcomes that can be translated into monetary payouts).

2.2 Certainty equivalent and risk premium

Definition of the certainty equivalent

Definition 2.1: Certainty equivalent

- The **certainty equivalent**, $\hat{x}$, of a lottery $\mathbf{L}$ is a certain payment that will make an individual indifferent between getting the lottery $\mathbf{L}$ and getting its certainty equivalent $\hat{x}$.
- If u is the individual's utility function, then:

$$u(\hat{x}) = \mathrm{E}[u(\mathbf{L})].$$

Comment:

- Note that, ceteris paribus, the greater an individual's risk aversion, the lower will be $\hat{x}$.

Definition of the equivalent risk premium

Definition 2.2: Equivalent risk premium

- The equivalent **risk premium**, r, for an individual facing lottery **L** is defined as

$$r = \bar{x} - \hat{x},$$

- where $\bar{x} \equiv E(\mathbf{x}) \equiv E(\mathbf{L})$.

Comments:

- The **equivalent** risk premium measures how much certain income an individual is **willing to pay to avoid a given risk**.
- Alternatively, the **compensating** risk premium measures how much certain income would be necessary to make an individual willing to **accept a given risk**.

- In this course, we will only use the equivalent risk premium and simply refer to it as **risk premium**.
- Note that, ceteris paribus, the greater an individual's risk aversion, the higher will be r.
- This implies that the risk premium can be used as a **first crude measure for risk aversion:**
  - Individual A is more risk-averse than individual B ...
  - ... if, for the same risk, A is willing to pay more than B to get rid of this risk ...
  - ... i.e. if $r_A > r_B$.

A first (crude) measure of risk aversion

Theorem 2.1: Risk premium and risk aversion

1. The **risk premium is positive** iff an individual is **risk-averse.**
2. The **risk premium is negative** iff an individual is **risk-loving.**
3. The **risk premium is zero** iff an individual is **risk-neutral.**

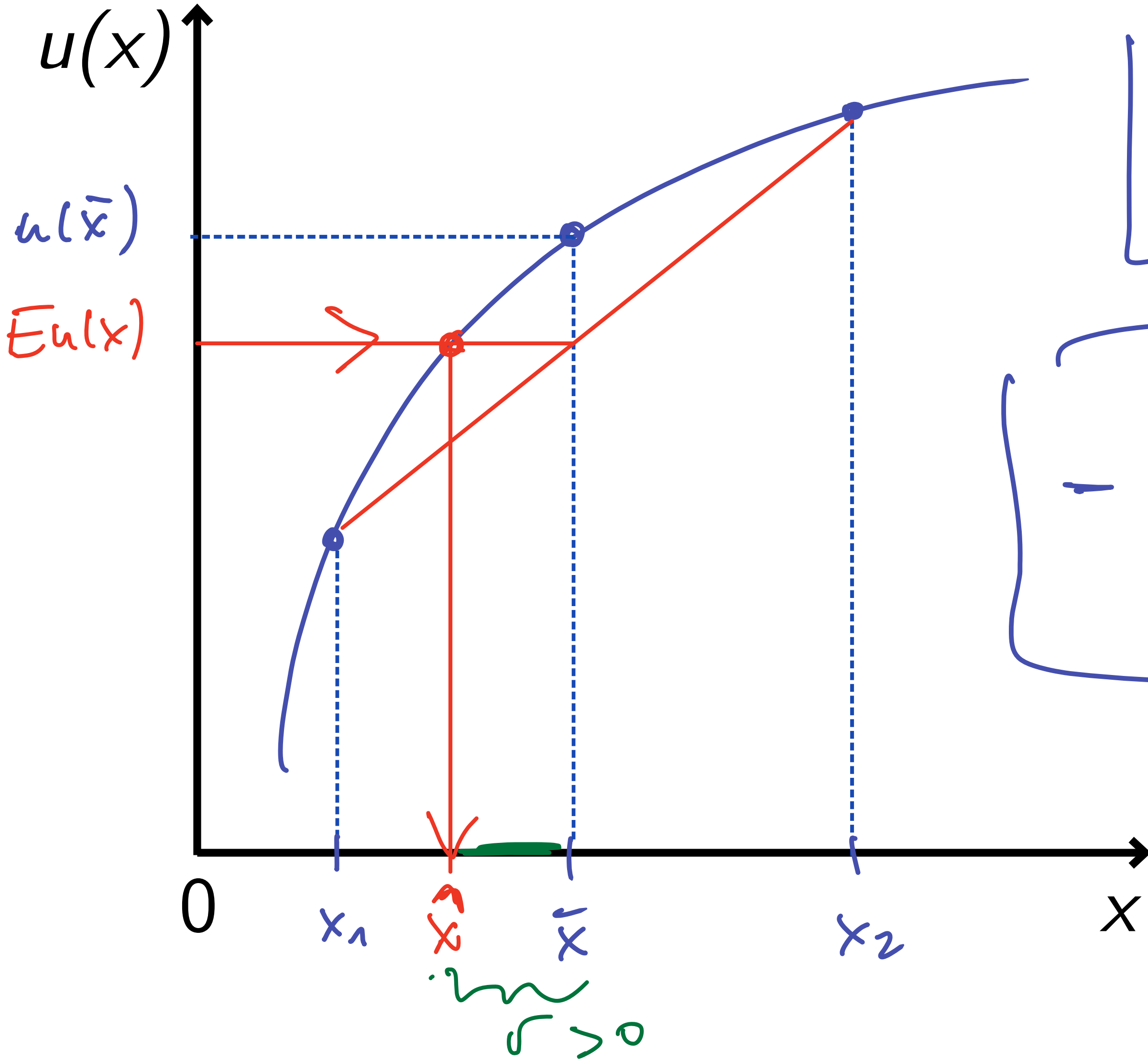
Proof:

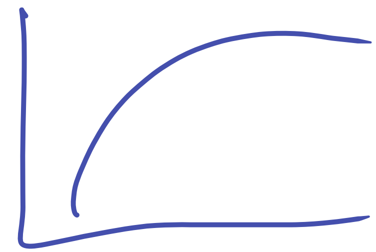
- By definition: $r = \bar{x} - \hat{x}$. [1]
- Also by definition: $u(\hat{x}) = \mathrm{E}[u(\mathbf{L})]$. [2]
- Jensen's Inequality: $\mathrm{E}[u(\mathbf{L})] < u(\mathrm{E}[\mathbf{L}])$ iff u is strictly concave. [3]
- From [1] follows: $r > 0 \iff \bar{x} > \hat{x} \iff u(\bar{x}) > u(\hat{x})$. (since $u' > 0$ by hypothesis)

- From [2] follows:
  $u(\bar{x}) > u(\hat{x}) \iff u(\bar{x}) > \mathrm{E}[u(\mathbf{L})] \iff u(\mathrm{E}[\mathbf{L}]) > \mathrm{E}[u(\mathbf{L})]$.
  (since $\bar{x} \equiv \mathrm{E}[\mathbf{L}]$)
- From [3] follows: $r > 0 \iff u$ strictly concave.
- QED.
- The proofs for sentences 2 and 3 of theorem 2.1 follow the same spirit and are implied by the definitions of strict convexity and linearity, respectively (see. chapter 1).

Graph: Certainty equilibrium and risk premium

$\hat{x}: \ u(\hat{x}) = Eu(x)$





$$-\frac{u''(x)}{u'(x)}$$

$u''(x) < 0$, $u'(x) > 0$

2.3 Local measures of risk aversion

Motivation

- Our goal is to find a measure of risk aversion that allows for **interpersonal comparisons.**
- Given that a concave utility function is the precondition for risk aversion, it is intuitive that the degree of risk aversion depends on **"how strongly concave"** the utility function is.
- Since the concavity of the utility function depends on its second derivative, one might believe that u'' is a good candidate for such a measure.
  - $-u''_A(\mathbf{x}) > -u''_B(\mathbf{x}) \iff A$ is more risk-averse then B.
- However, ordinal utility functions are **only unique up to positive linear transformations** which renders the second derivative an unreliable measure.

- To see this, consider the following **example:**
  - Let $u(\mathbf{x})$ and $v(\mathbf{x}) = \alpha + \beta u(\mathbf{x})$ define the same preferences.
  - Recall the definition of the certainty equivalent: $u(\hat{x}) = \mathrm{E}[u(\mathbf{x})]$.
  - Then: $\mathrm{E}[v(\mathbf{x})] = \alpha + \beta \mathrm{E}[u(\mathbf{x})] = \alpha + \beta u(\hat{x}) = v(\hat{x})$.
  - This means that $\hat{x}$ is the certainty equivalent for both utility functions.
  - It follows that the risk premia are also the same for both u and v.
  - However, $v''(\mathbf{x}) = \beta u''(\mathbf{x}) \neq u''(\mathbf{x})$ if $\beta \neq 1$.
  - So, while both utility functions describe the same risk preferences, the second derivatives would (falsely) suggest different degrees of risk aversion.
- But what if we **normalized** our naïve measure?
  - How about $-\frac{v''(\mathbf{x})}{v'(\mathbf{x})} = -\frac{\beta u''(\mathbf{x})}{\beta u'(\mathbf{x})} = -\frac{u''(\mathbf{x})}{u'(\mathbf{x})}$?
  - This looks pretty good ... and it is: $\rightarrow$ Pratt-Arrow coefficient of absolute risk aversion (next slide).

The Pratt-Arrow coefficient of absolute risk aversion

Definition 2.3: Pratt-Arrow coefficient of absolute risk aversion

- Let $u(\mathbf{x})$ be an at least twice differentiable utility function, and let w denote initial wealth.
- The **Pratt-Arrow coefficient of absolute risk aversion** $(A(w))$ is defined as:

$$A(w) \equiv -\frac{u''(w)}{u'(w)}.$$

Comments:

- One can show (with a 2nd-order Taylor approximation) that $A(w)$ is an **approximation of the risk premium:** $r_x \approx -\frac{u''(w)}{u'(w)}\frac{\sigma_x^2}{2}$.
- **Risk tolerance**, $T(w)$, is defined as $T(w) = \frac{1}{A(w)}$.

A(w)

The Pratt-Arrow coefficient of relative risk aversion

Definition 2.4: Pratt-Arrow coefficient of relative risk aversion

- Let $u(\mathbf{x})$ be an at least twice differentiable utility function, and let w denote initial wealth.
- The **Pratt-Arrow coefficient of relative risk aversion** $(R(w))$ is defined as:

$$R(w) \equiv -w \frac{u''(w)}{u'(w)}.$$

Comments:

- One can show (2nd-order Taylor approximation) that $R(w)$ is an **approximation of the relative risk premium:** $\rho_x \approx -w \frac{u''(w)}{u'(w)} \frac{\sigma_x^2}{2}$
- Note that $R(w) = w \cdot A(w)$.

Absolute vs. relative risk aversion

- Why do we need two measures for risk aversion and when do we use which?
- This depends on the question we want to study.
- **Absolute risk aversion** measures risk preferences over **absolute amounts of wealth.**
  - Lottery over final wealth: $\mathbf{w}_i = w + \mathbf{x}_i$.
  - E.g.: How high is my risk aversion towards a gamble over 10 EUR?
- **Relative risk aversion** measures risk preferences over **a certain percentage of wealth.**
  - Lottery over final wealth: $\mathbf{w}_i = w(1 + \mathbf{x}_i)$.
  - E.g.: How high is my risk aversion towards a gamble over 1% of my wealth?
- The different usage of $A(w)$ and $R(w)$ will become especially clear when studying the **comparative statics** of both measures with respect to wealth.

A real world example: Cross-country differences in speeding

Swede faces world-record $1m speeding penalty

12 August 2010 | Europe

A Swedish motorist caught driving at 290km/h (180mph) in Switzerland could be given a world-record speeding fine of SFr1.08m ($1m; £656,000), prosecutors say.

The Swede was driving a Mercedes SLS AMG - which has a top speed of 317km/h

The 37-year-old, who has not been named, was clocked driving his Mercedes sports car at 170km/h over the limit.

Under Swiss law, the level of fine is determined by the wealth of the driver and the speed recorded.

In January, a Swiss driver was fined $290,000 - the current world record.

Local police spokesman Benoit Dumas said of the latest case that "nothing can justify a speed of 290km/h".

"It is not controllable. It must have taken 500m to stop," he said.

Comparative-static properties of $A(w)$

Definition 2.5: DARA, CARA, and IARA

An at least twice differentiable utility function has the property of ...

1. ... **decreasing absolute risk aversion (DARA)** iff $\frac{dA(w)}{dw} < 0$.
2. ... **constant absolute risk aversion (CARA)** iff $\frac{dA(w)}{dw} = 0$.
3. ... **increasing absolute risk aversion (IARA)** iff $\frac{dA(w)}{dw} > 0$.

Examples:

1. DARA: The richer I am, the less risk-averse I am to bet 10 EUR.
2. CARA: If I become richer, my risk aversion to bet 10 EUR stays the same.
3. IARA: The richer I am, the more risk-averse I am to bet 10 EUR.

Comparative-static properties of $R(w)$

Definition 2.6: DRRA, CRRA, and IRRA

An at least twice differentiable utility function has the property of ...

1. ... **decreasing relative risk aversion (DRRA)** iff $\frac{dR(w)}{dw} < 0$.
2. ... **constant relative risk aversion (CRRA)** iff $\frac{dR(w)}{dw} = 0$.
3. ... **increasing relative risk aversion (IRRA)** iff $\frac{dR(w)}{dw} > 0$.

Examples:

1. DRRA: The richer I am, the less risk-averse I am to bet 1% of my income.
2. CRRA: If I become richer, my risk aversion to bet 1% of my income stays the same.
3. IRRA: The richer I am, the more risk-averse I am to bet 1% of my income.

What are realistic comparative-static properties?

- The following assumptions are commonly held to be most plausible:
  - **Absolute** risk aversion: **DARA** ($\frac{dA(w)}{dw} < 0$)
  - **Relative** risk aversion: **CRRA or IRRA** ($\frac{dR(w)}{dw} \geq 0$)
- While the intuition for DARA is straightforward (beggar vs. millionaire having to risk 10 EUR), the assumptions for CRRA/IRRA merit some discussion.
  - Recall that $R(w) = wA(w)$.
  - The chain rule implies that the overall effect of changing wealth on $R(w)$ can be decomposed into **two separate effects:**

$$\frac{dR(w)}{dw} = A(w) + w\frac{dA(w)}{dw}.$$

- Let us start with the **first effect:**
  - It will be positive (given risk aversion): $A(w) > 0$
  - Intuition: As wealth increases, the absolute amount of money that is put at risk, increases, too (e.g.: 1% of wealth).
  - Since the individual does not like risk ($A(w) > 0$) she will be less willing to risk a larger amount than a smaller amount.
- Now the **second effect:**
  - For DARA utility ($\frac{dA(w)}{dw} < 0$), it will be negative.
  - Intuition: As wealth increases, absolute risk aversion decreases.
  - As a consequence, risking such a large amount of money (in absolute terms) is not as painful as if the individual were poorer.
  - This (partly) offsets the first effect.
- **Overall effect:**
  - It is commonly assumed that the second effect is smaller or equal to the first effect (in absolute terms), but typically not larger.
  - Hence, $\frac{dR(w)}{dw} \geq 0$.

Properties of some typical utility functions

1. **Quadratic utility** (CAPM)
   - $u(w) = w - \alpha w^2$
   - Only realistic ($u'(w) > 0$) for limited support. But popular, because easy to handle and in line with $\mu - \sigma$ criterion.
   - **IARA** (Verify this yourself!)
   - **IRRA** (Follows automatically. Why?)
2. **Logarithmic utility**
   - $u(w) = \ln(w)$
   - One of the simplest work horses for modeling utility functions.
   - **DARA** (Verify this yourself!)
   - **CRRA** (Verify this yourself!)
3. **Power utility**
   - $u(w) = \frac{w^{1-\sigma}}{1-\sigma}$
   - Popular due to its flexibility. We focus on $\sigma > 0$ ($\rightarrow$ concavity).
   - **DARA** (Verify this yourself!)
   - **CRRA** (Verify this yourself!)

4 Exponential utility

- $u(w) = -e^{-\alpha w}$
- Also commonly used. We focus on $\alpha > 0$ ($\rightarrow$ concavity).
- **CARA** (Verify this yourself!)
- **IRRA** (Follows automatically. Why?)

5 Hyperbolic utility

- $u(w) = \frac{1-\gamma}{\gamma}\left(\frac{\alpha w}{1-\gamma} + \beta\right)^{\gamma}$, with $\frac{\alpha w}{1-\gamma} + \beta \geq 0$, and $\alpha > 0$.
- Also popular due to its flexibility.
- Can be all: **DARA/CARA/IARA,** depending on the value of γ (Verify this yourself!)
- Can be all: **DRRA/CRRA/IRRA,** depending on the values of β and γ (Verify this yourself!)
- In any case: **HARA:** $A(w)$ is a hyperbolic function in w: $A(w) = \frac{\alpha}{\frac{\alpha w}{1-\gamma} + \beta}$.
- This implies that risk tolerance, $T(w) = \frac{1}{A(w)}$ is a linear function in w, which can be a desired property in certain modeling contexts: $T(w) = \frac{w}{1-\gamma} + \frac{\beta}{\alpha}$

Caution: So far only local measure of risk aversion

- The Pratt-Arrow coefficients of absolute and relative risk aversion are only **local measures of risk preferences.**
- This is, they are only valid in the neighborhood of the initial wealth level, w that is under consideration.
- For other wealth levels, results could well be reversed.
  - E.g.: Interpersonal comparison of risk aversion:
  - For low wealth levels: A is more risk-averse than B.
  - For high wealth levels: B is more risk-averse than A.
- If we want to make **global statements** (over the entire range of w), we will have to make stricter assumptions **(next subchapter).**

2.4 Global measures of risk aversion

Theorem 2.2: Global measures of risk aversion

Consider two individuals with utility functions, u_A and u_B, and a given lottery $\mathbf{L}$ that adds a risky component, $\mathbf{x}$ to initial wealth w. **The following statements are equivalent:**

1. A is globally more risk averse than B.
2. $A_A(w) \geq A_B(w) \quad \forall w$
3. $r_A(\mathbf{x}, w) \geq r_B(\mathbf{x}, w) \quad \forall w, \mathbf{x}$
4. $u_A(\cdot) = G(u_B(\cdot))$, with G being a concave function.
5. $\frac{u_A(w_3)-u_A(w_2)}{u_A(w_1)-u_A(w_0)} \leq \frac{u_B(w_3)-u_B(w_2)}{u_B(w_1)-u_B(w_0)} \quad \forall w_0 < w_1 \leq w_2 < w_3$

Comments:

- The intuition of this theorem should be clear.
- We will not prove the entire theorem here (but maybe look at some of its components in one of the next tutorials).

2.5 Even more measures of risk preferences

- The buck did not stop in the 1960s.
- Besides $A(w)$ and $R(w)$, the literature developed more measures.
- Let us just browse through a few:
- **Partial risk aversion** (Pratt, 1960; Arrow, 1970; Ross, 1981)
  - Idea: Like relative risk aversion, but the gamble is only over a certain part (w_1) of initial wealth. The other part (w_0) is secure: $\mathbf{w}_i = w_0 + w_1(1 + \mathbf{x}_i)$
  - Measure of partial risk aversion: $R_p(w) \equiv -w_1 \frac{u''(w_0+w_1)}{u'(w_0+w_1)}$.
- **Absolute prudence** (Kimball, 1990)
  - Idea: Risk as an intertemporal problem: Future consumption is uncertain. Prudent individuals will sacrifice current consumption and increase precautionary savings, $s = w - c$.
  - Measure of absolute prudence: $p_x(w - c) \equiv -\frac{u'''(w-c)}{u''(w-c)}$

2.5 Even more measures of risk preferences

- For a given wealth level: The higher $p_x(w-c)$, the less the individual will consume in the present (in absolute terms), and, hence, the more she will save for the future (in absolute terms).
- Common assumption: Prudence decreases in wealth: $\frac{dp_x(w-c)}{dw} < 0$.
  - As w increases, present consumption increases (in absolute terms).
  - However, the effect on the absolute level of savings is unclear: $s = w \uparrow - c \uparrow$.

- **Absolute temperance** (Kimball, 1992; Gollier and Pratt, 1996)
  - Idea: Even higher-order risk attitudes (4th derivative!) have implications in a wide range of economic applications, such as bargaining, bidding in auctions, rent seeking, sustainable development, tax compliance, valuation of medical treatments, etc.
  - Measure of absolute temperance: $t(w) \equiv -\frac{u''''(w)}{u'''(w)}$.
- **Practical relevance** of such high-order risk attitudes is **questionable**, especially in light of likely dominating behavioural biases