Economic Foundations and Applications of Risk

Part A. Foundations

Chapter 3: Measures of Risk

Till Stowasser

LMU, 2023

Syllabus

- 3.1 Introduction
- 3.2 First-order stochastic dominance
- 3.3 Second-order stochastic dominance

3.1 Introduction

- In the previous chapter we learnt how to rank individuals according to their risk aversion.
- In this chapter, we will study **how to rank monetary lotteries with respect to** their riskiness, and ultimately, **their desirability.**
- After a motivational example and a short refresher on integration by parts . . .
- . . . we will introduce the concepts of **first-order stochastic dominance** (3.2) and **second-order stochastic dominance** (3.3).

Motivational example

- Which lottery would a risk-averse individual with $u(x) = \sqrt{x}$ prefer?
  - $\mathbf{L}_1 = (\frac{7}{8}, \frac{1}{8}; 1, 9)$, or
  - $\mathbf{L}_2 = (\frac{1}{2}, \frac{1}{2}; 0, 4)$
- Idea: Use variance as raw measure for risk:
  - Expected values: $\mathrm{E}[\mathbf{L}_1] = 2 = \mathrm{E}[\mathbf{L}_2]$
  - Variances: $\mathrm{Var}[\mathbf{L}_1] = 7 > 4 = \mathrm{Var}[\mathbf{L}_2]$
- So, risk-averse individual should prefer $\mathbf{L}_2$, right?
- Well, she does not!
  - $\mathrm{E}[u(\mathbf{L}_1)] = \frac{7}{8} \cdot \sqrt{1} + \frac{1}{8} \cdot \sqrt{9} = \frac{10}{8}$
  - $\mathrm{E}[u(\mathbf{L}_2)] = \frac{1}{2} \cdot \sqrt{0} + \frac{1}{2} \cdot \sqrt{4} = 1$
  - $\Longrightarrow \mathbf{L}_1 \succ \mathbf{L}_2$
- We need a better measure than just the expected value and the variance of a lottery.

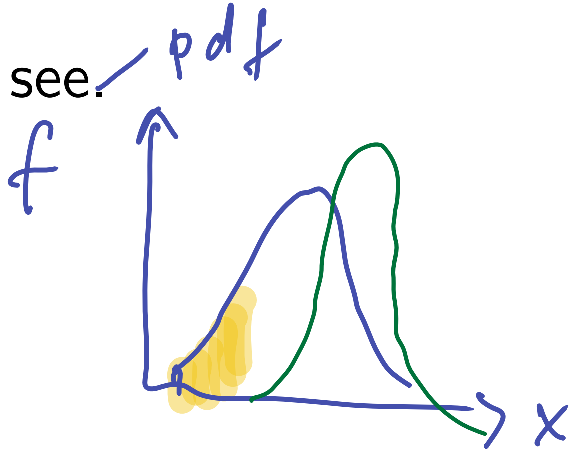
Stochastic Dominance (SD)

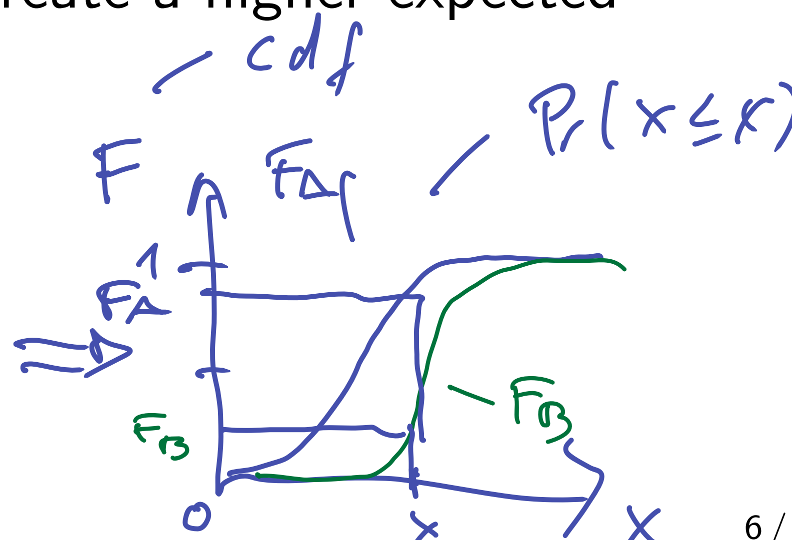
- SD is a concept that allows a preference ranking of distributions.
- While satisfying the property of **transitivity**, this concept is **not complete**, i.e. it will not be possible to rank all distributions.
- **First-order stochastic dominance (FOSD)**. . .
  - . . . is a **very general** measure that allows a preference ordering for all utility functions u with $u' > 0$.
  - Its downside is that the **ranking is very incomplete.**
- **Second-order stochastic dominance (SOSD)**. . .
  - . . . is **less general,** as it only holds for risk-averse individuals with $u'' < 0 < u'$.
  - It allows for a **less incomplete ranking** than FOSD, even though there will still be lotteries that cannot be generally ranked by SOSD either.
- There are concepts of **higher-order stochastic dominance,** which allow for the ranking of a vaster class of distributions, but which, in turn, require starker restrictions on utility function u.

3.2 First-order stochastic dominance

A simple question

- Let there be two distributions A and B, described by their cumulative distribution functions (CDF), F_A and F_B, respectively.
- f_A and f_B are the respective densities, which exist by hypothesis (i.e. we assume the CDFs to be continuously differentiable).
- **Question:** When will distribution B create a higher expected utility than distribution A?
- **Answer:** Let's see.





Definition of first-order stochastic dominance

Definition 3.1: First-order stochastic dominance

- Let $F_A(x)$ and $F_B(x)$ be two continuously differentiable cumulative distribution functions.
- Then F_B is said to **first-order stochastically dominate** F_A iff

$$\forall x \in \mathbb{R} : F_B(x) \leq F_A(x)$$

and

$$\exists x \in \mathbb{R} : F_B(x) < F_A(x).$$

Comments:

- Recall that $F_B(x) \leq F_A(x) \equiv Prob_B(X \leq x) \leq Prob_A(X \leq x)$
- **Intuition:** Distribution B always has a lower probability to return a lower x than distribution A.

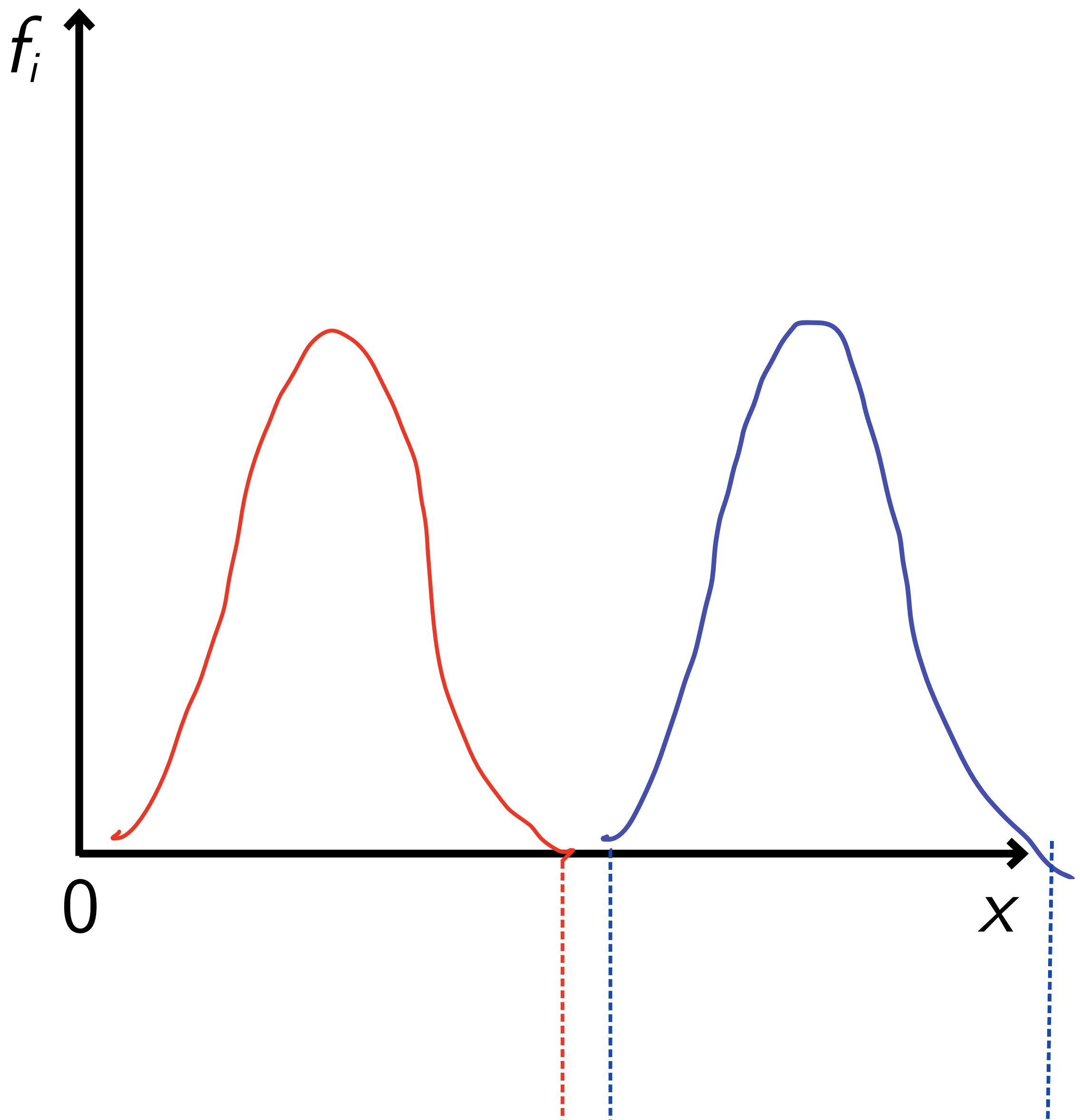
FOSD theorem

Theorem 3.1: FOSD theorem

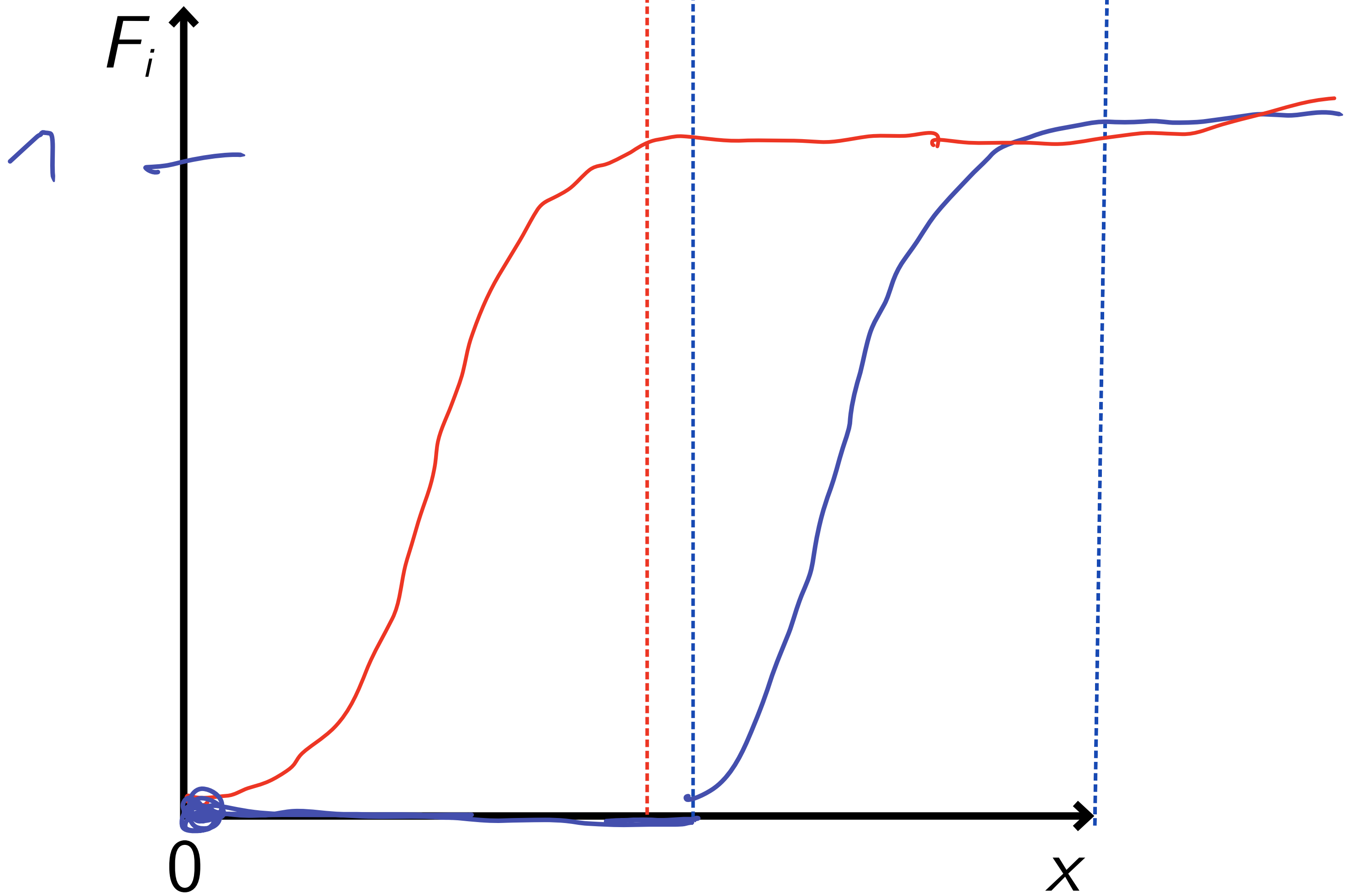
1. Risk-loving, risk-neutral, and risk-averse individuals with a **positive marginal utility** in income **prefer the first-order stochastically dominating distribution** of income.
2. This implies that FOS dominated distributions have a lower **expected value** than FOS dominating distributions (Necessary, but not sufficient condition for FOSD).

$$F_B(x) \succ^{FOSD} F_A(x) \Longrightarrow \mathrm{E}_B[x] > \mathrm{E}_A[x]$$

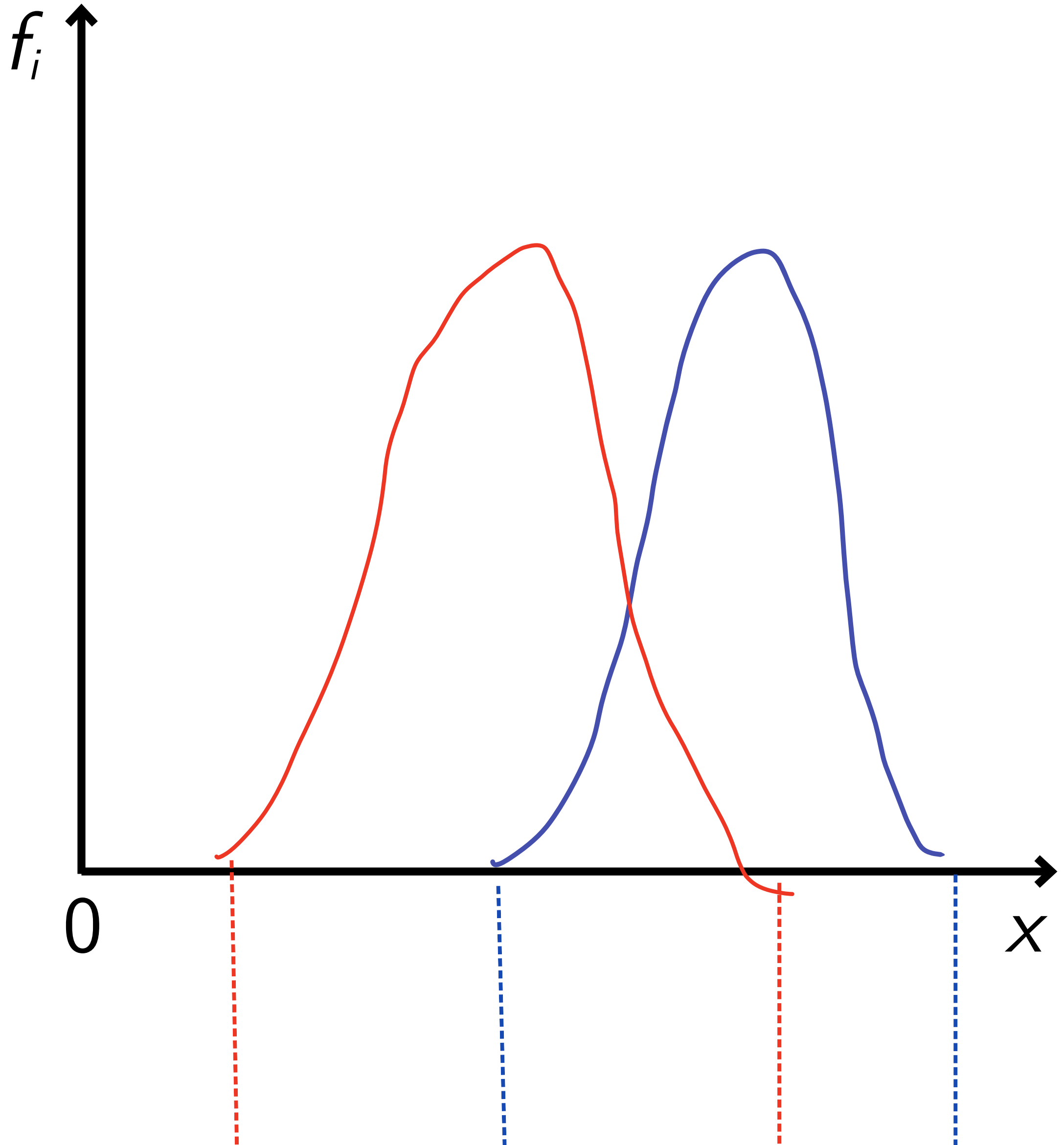
Example for strict domination: Densities



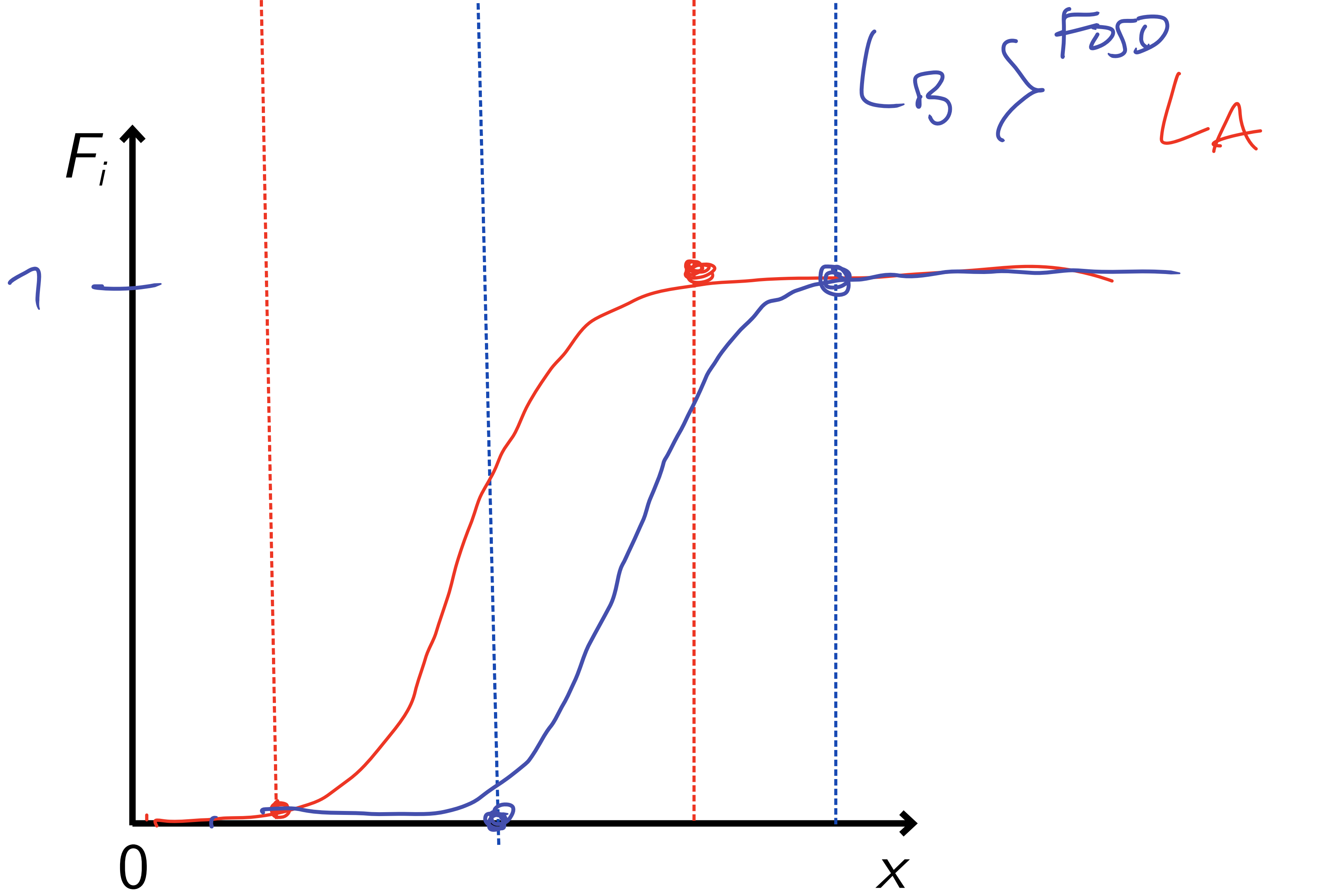
Example for strict domination: CDFs



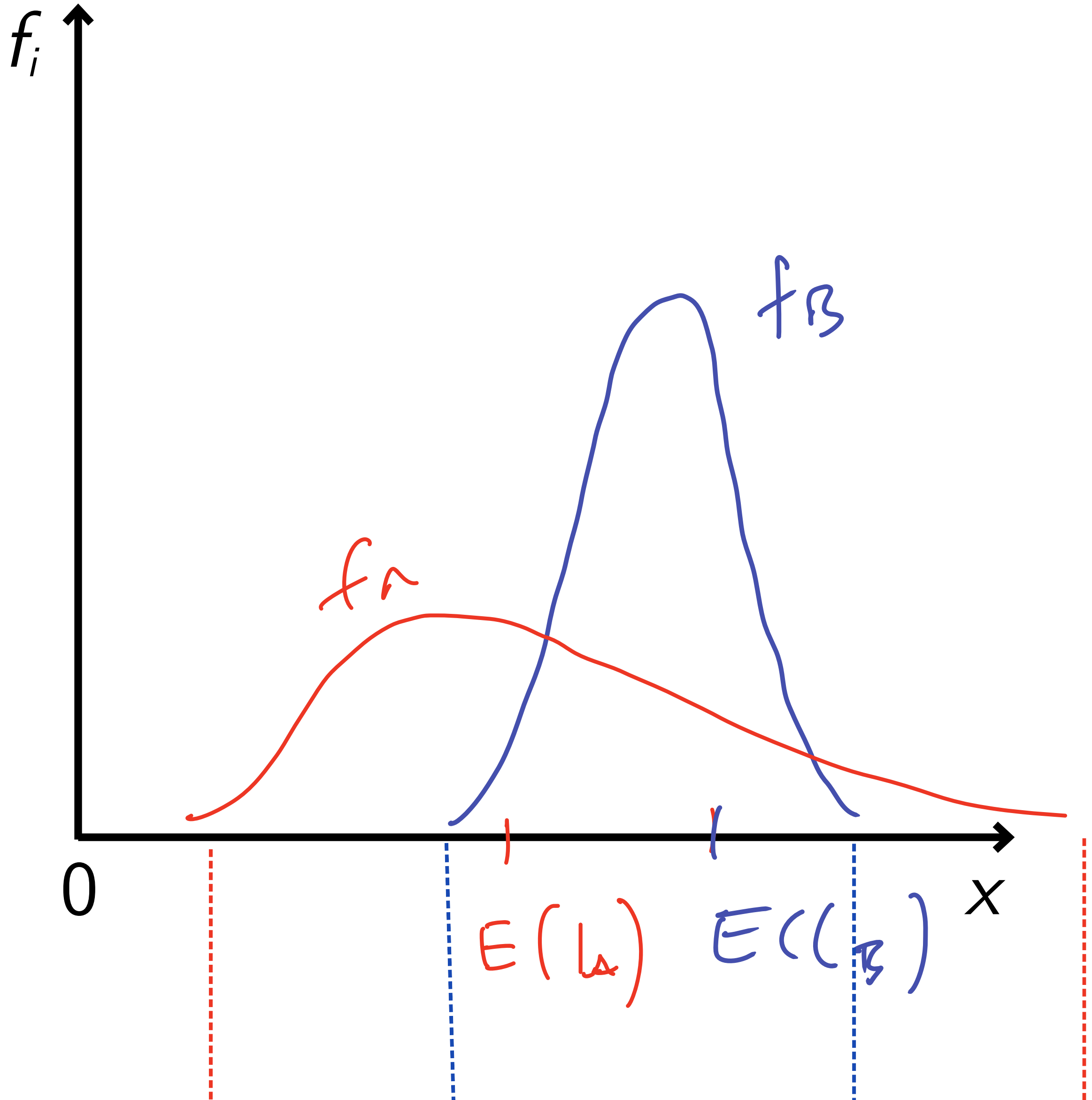
Example for FOS domination: Densities



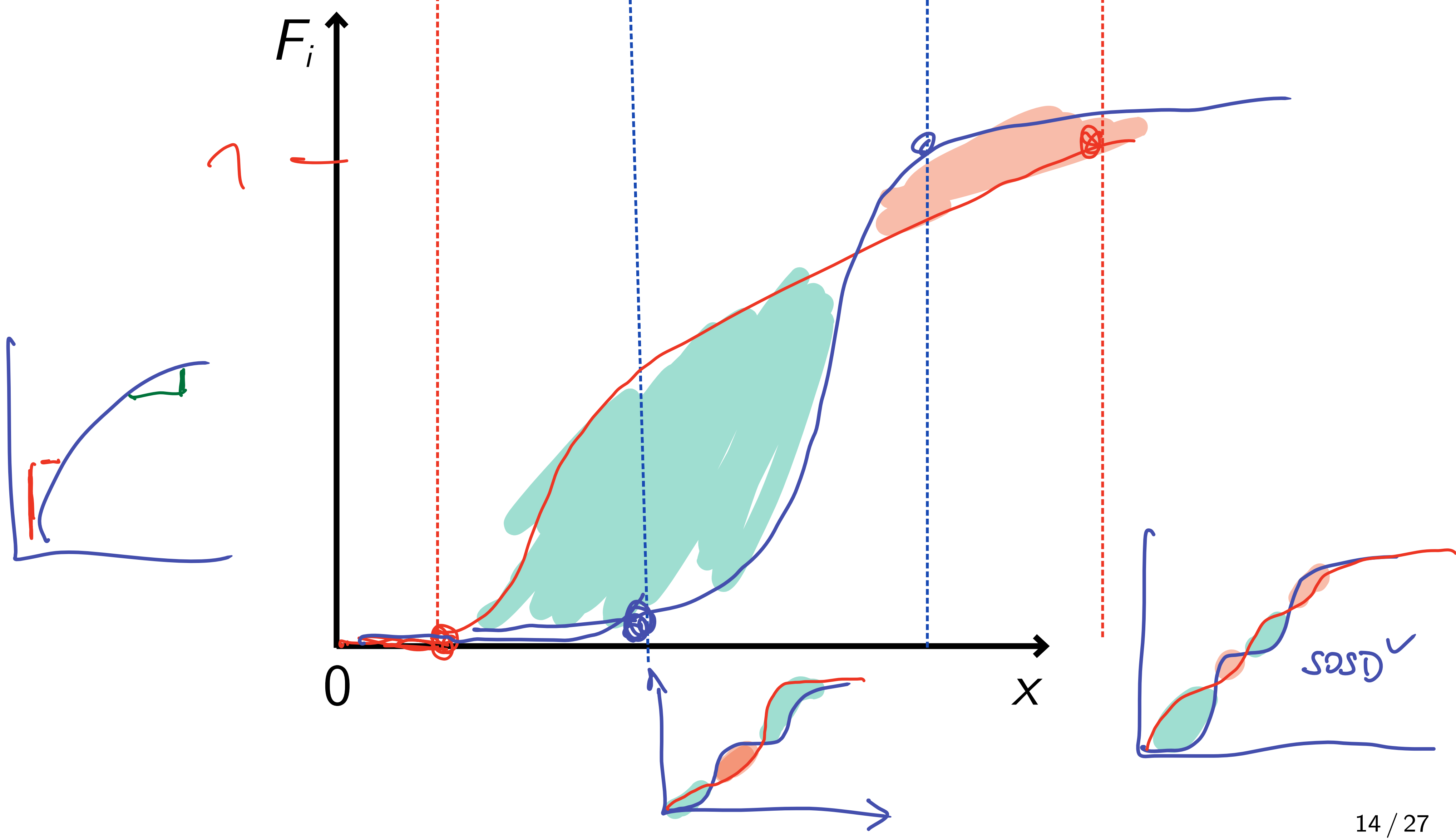
Example for FOS domination: CDFs



Example for FOS non-domination: Densities



Example for FOS non-domination: CDFs



3.3 Second-order stochastic dominance

Definition of second-order stochastic dominance

Definition 3.2: Second-order stochastic dominance

- Let $F_A(x)$ and $F_B(x)$ be two continuously differentiable cumulative distribution functions.
- Then F_B is said to **second-order stochastically dominate** F_A iff

$$\forall x \in [a; b] : \int_a^b F_B(x)dx \leq \int_a^b F_A(x)dx$$

and

$$\exists x \in \mathbb{R} : \int_a^b F_B(x)dx < \int_a^b F_A(x)dx.$$

Comments:

- Note that the definition of SOSD is basically the same as that for FOSD but with integrals.
- For FOSD, distribution B needs to be better (in expected value) than distribution A for any value of x.
  - The CDFs are not allowed to cross.
- For SOSD, distribution B only needs to have an expected **cumulated** advantage over distribution A for any value of x.
  - The CDFs are allowed to cross.
  - More specifically: A is allowed to be better (in expected value) for high values of x, as long as the advantage of B for lower values of x is not more than fully compensated.
  - Reason: Ask yourselves, what could be the intuition for this result?

Comments:

- Note that the definition of SOSD is basically the same as that for FOSD but with integrals.
- For FOSD, distribution B needs to be better (in expected value) than distribution A for any value of x.
  - The CDFs are not allowed to cross.
- For SOSD, distribution B only needs to have an expected **cumulated** advantage over distribution A for any value of x.
  - The CDFs are allowed to cross.
  - More specifically: A is allowed to be better (in expected value) for high values of x, as long as the advantage of B for lower values of x is not more than fully compensated.
  - Reason: Risk aversion implies that the advantage for low x is more important than the disadvantage for high x.

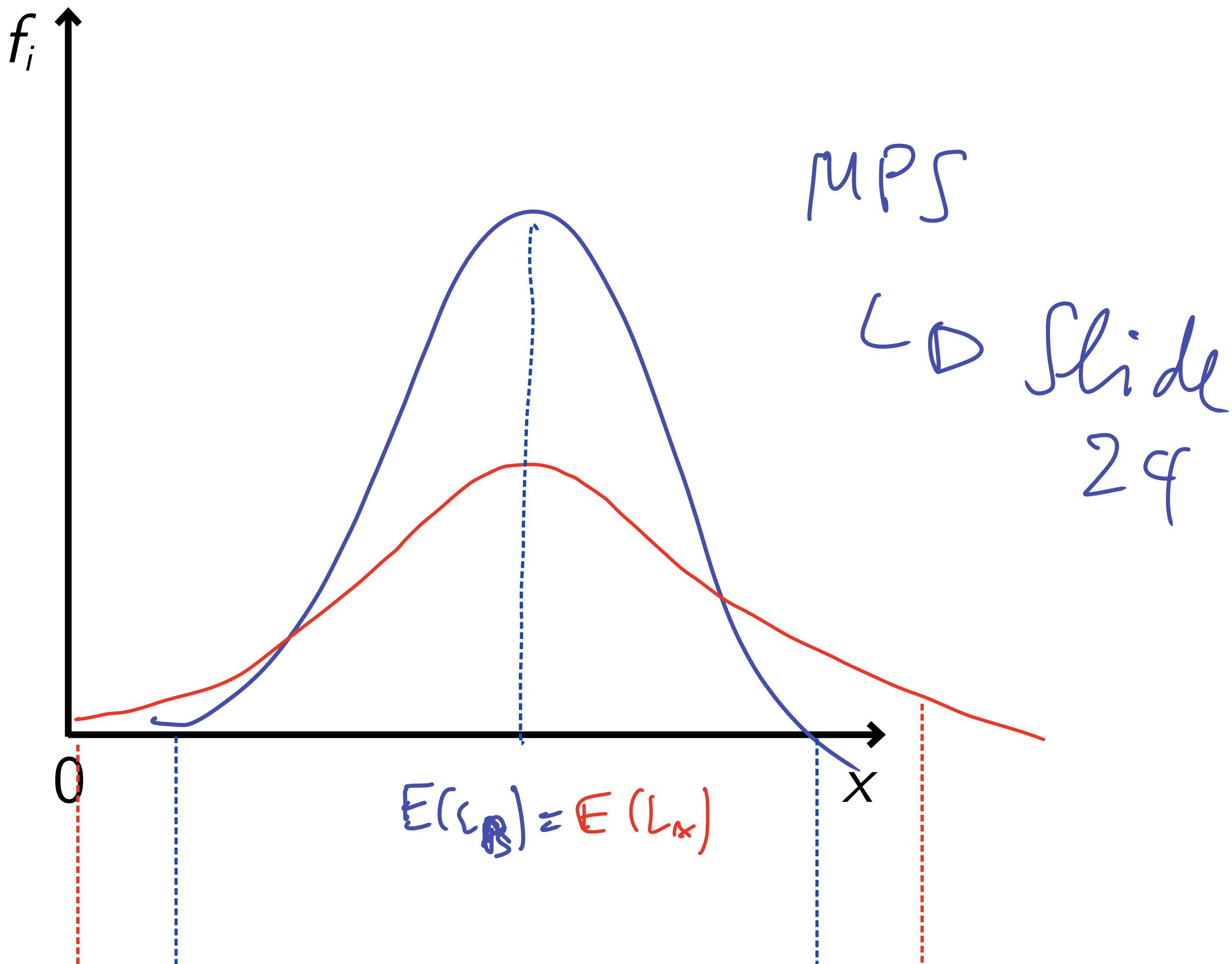
SOSD theorem

Theorem 3.2: SOSD theorem

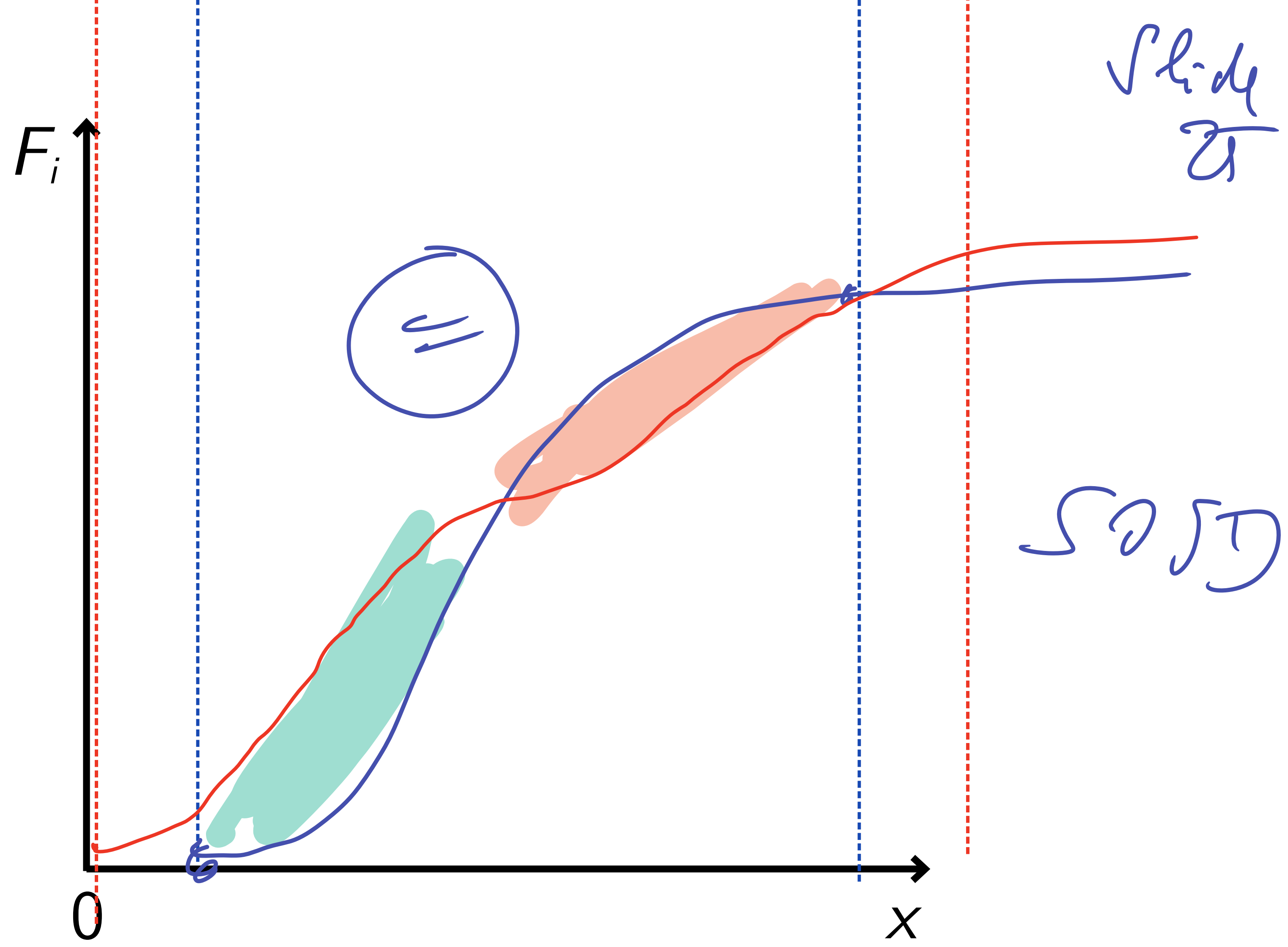
1. **Risk-averse** individuals with a positive marginal utility in income **prefer the second-order stochastically dominating distribution** of income.
2. This implies that SOS dominated distributions do not have a **higher expected value** than SOS dominating distributions (Necessary, but not sufficient condition for SOSD).

$$F_B(x) \succ^{SOSD} F_A(x) \implies \mathrm{E}_B[x] \geq \mathrm{E}_A[x]$$

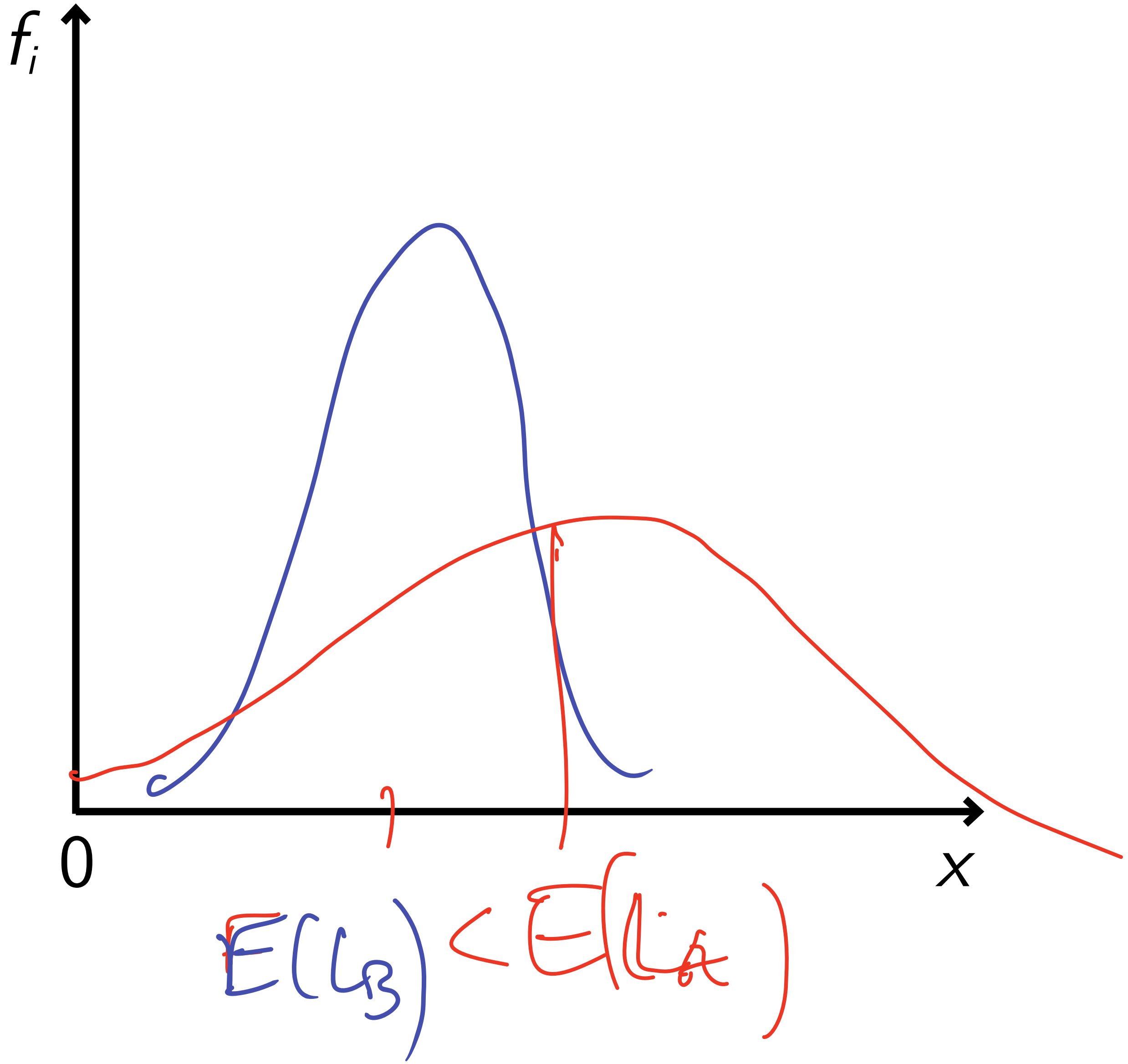
Example for SOS domination: Densities



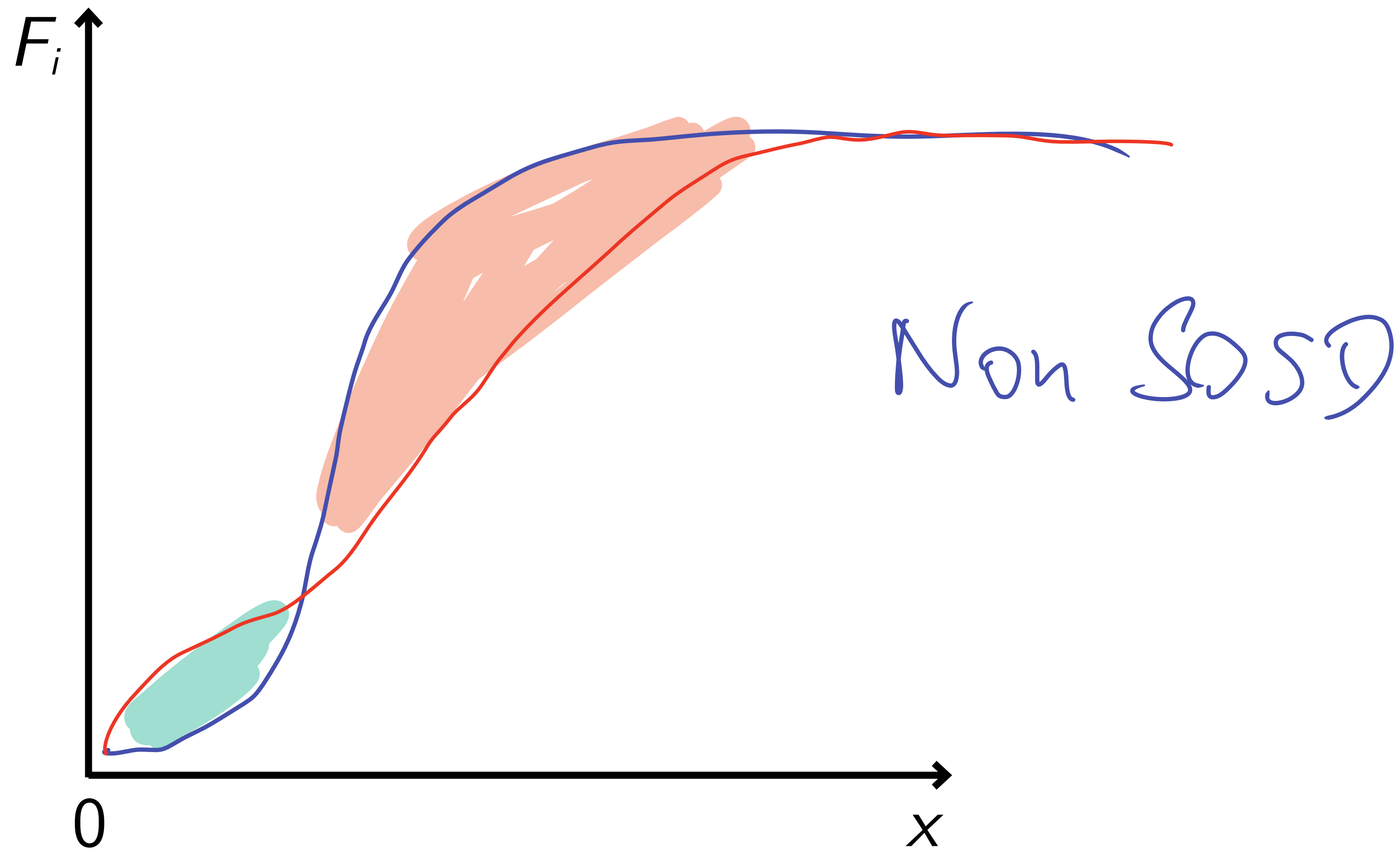
Example for SOS domination: CDFs



Example for SOS non-domination: Densities



Example for SOS non-domination: CDFs



Mean-preserving spread

Definition 3.3: Mean-preserving spread

$F_A(x)$ is said to be a **mean-preserving spread (MPS)** of $F_B(x)$ iff

$$F_B \succ^{SOSD} F_A$$

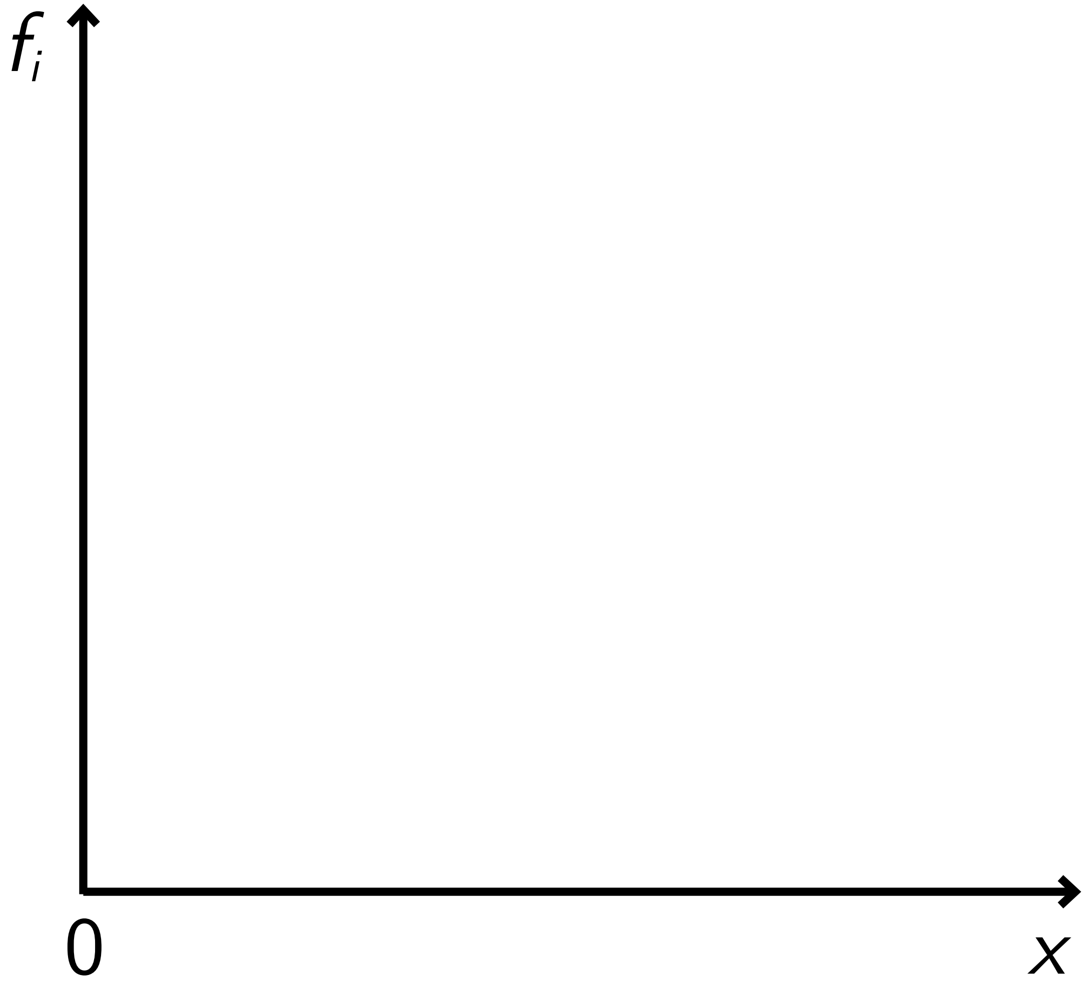
and

$$\mathrm{E}_A(x) = \mathrm{E}_B(x).$$

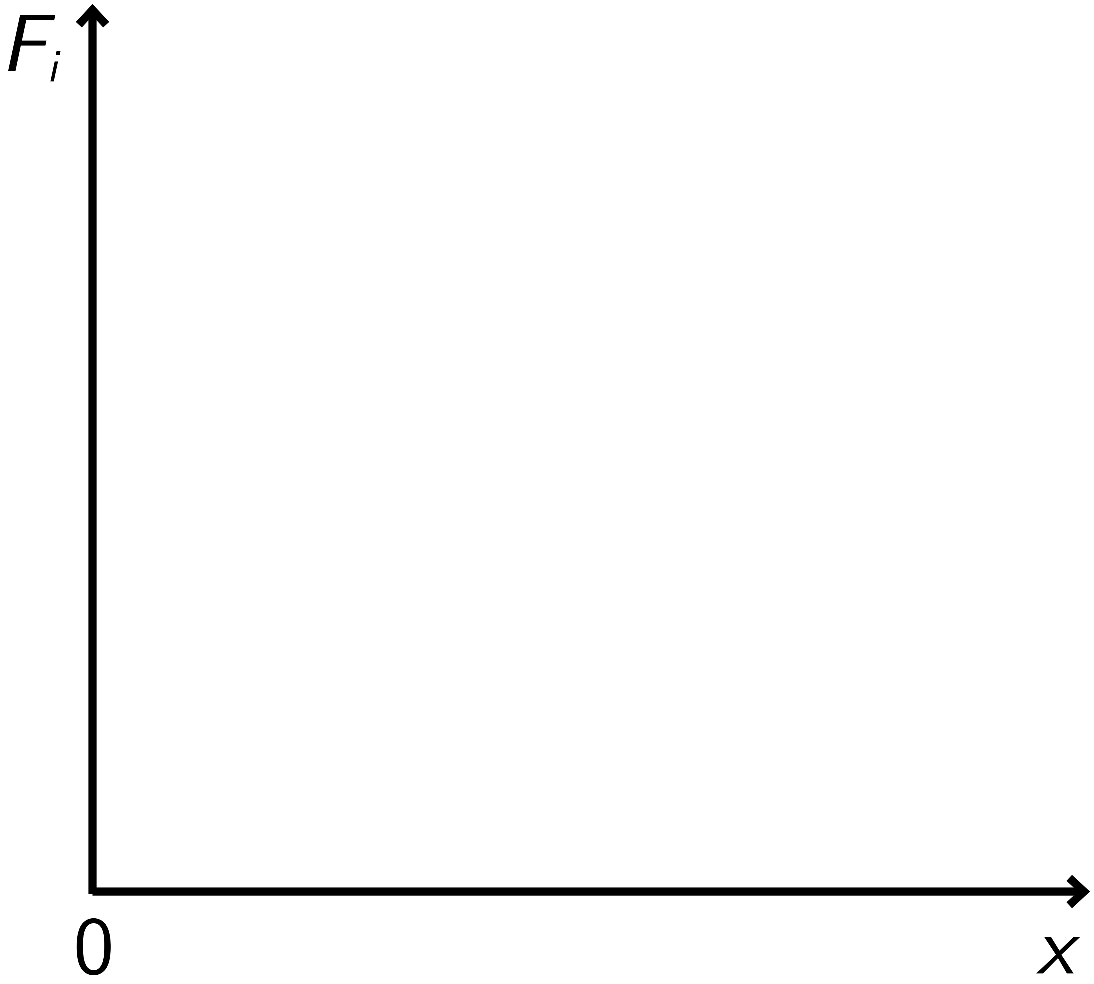
Comments:

- Note that a MPS is the border case between SOSD and non-SOSD.
- If $F_A(x)$ is a MPS of $F_B(x)$, then $F_A(x)$ has a higher variance than $F_B(x)$.

Example for MPS: Densities



Example for MPS: CDFs



The Rothschild-Stiglitz theorem (1970)

The Rothschild-Stiglitz theorem

Let there be two lotteries over $x \in [a;b]$, $\mathbf{L}_A$ and $\mathbf{L}_B$, with $\mathrm{E}_A(x) = \mathrm{E}_B(x)$. **The following statements are equivalent:**

1. Any and every risk-averse agent will prefer lottery $\mathbf{L}_B$ over $\mathbf{L}_A$.
2. $\forall x \in [a;b] : \int_a^x (F_A(u) - F_B(u))du \geq 0$.
3. $\mathbf{L}_A$ is a MPS of $\mathbf{L}_B$.
4. $\mathbf{L}_A$ is equal to $\mathbf{L}_B$ but for addition of white noise.

Comments:

- That [2] $\Rightarrow$ [1], we have already seen above. We will not prove [1] $\Rightarrow$ [2] here.
- [2] $\Leftrightarrow$ [3] is true by the very definition of MPS.
- [3] $\Leftrightarrow$ [4], because [4] is just a different way of describing a MPS.
- Let us prove [4] $\Rightarrow$ [1].

Proof of [4] $\Rightarrow$ [1]

- Let lottery $\mathbf{L}_A$ be defined over $y \in [a; b]$, and lottery $\mathbf{L}_B$ be defined over $x \in [a; b]$.
- Define white noise as ϵ: $y = x + \epsilon$, with $\mathrm{E}[\epsilon \mid x] = 0$
- Show that both distributions have the same mean
  - $\mathrm{E}_y[y] = \mathrm{E}_{x,\epsilon}[x + \epsilon] = \mathrm{E}_x[\mathrm{E}_\epsilon[(x + \epsilon) \mid x]] = \mathrm{E}_x[x]$
- Show that any risk-averse individual would prefer $\mathbf{L}_B$ over $\mathbf{L}_A$.
  - $\mathrm{E}_y[u(y)] = \mathrm{E}_{x,\epsilon}[u(x + \epsilon)] \iff$
  - $\mathrm{E}_y[u(y)] = \mathrm{E}_x[\mathrm{E}_\epsilon[u(x + \epsilon) \mid x]] \iff$
  - $\mathrm{E}_y[u(y)] < \mathrm{E}_x[u(x + \mathrm{E}_\epsilon[\epsilon \mid x])] = \mathrm{E}_x[u(x)] \iff$
  - $\mathbf{L}_A \prec \mathbf{L}_B$.
- QED.