Economic Foundations and Applications of Risk

Part B. Applications

Chapter 4: Optimal Portfolio Choice

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Syllabus

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4.1 Introduction

- One of the most archetypal decisions under risk is the **investment decision.**
- In this chapter, we will study a simple **portfolio choice model,** where an individual has to choose between a secure and a risky asset (4.2).
- We will concentrate on the **comparative static properties** of the investment decisions with respect to ...
  - ... the individual's **risk aversion** (4.3),
  - ... the **riskiness of the risky asset** (4.4),
  - ... an individual's **initial income** (4.5), and
  - ... the **return of the secure asset** (4.6).

4.2 Main model

Setup

- An individual can invest her initial wealth, w_0, in a **riskless asset** that pays out $(1 + i)$ with certainty ...
- ... and a **risky asset** with payout $(1 + \tilde{x})$ (where $\mu \equiv E[x]$).
- Let m denote the amount invested in the risk-free asset and a the sum invested in the risky asset.
- **Final wealth,** w, is then given by $w = m(1 + i) + a(1 + x)$.
- The **maximization problem** takes the following form:

$$\max_{a,m} \mathrm{E}[u(\underbrace{m(1+i) + a(1+x)}_{w})] \qquad \text{s.t. } m + a \leq w_0$$

$$\Leftrightarrow m = w_0 - a$$

$$\max_{a} E[u(w)] \qquad E[u(\underbrace{(w_0 - a)(1+i) + a(1+x)}))]$$

$$\Leftrightarrow \quad w_0(1+i) \cancel{-a} - ai \cancel{+a} + ax$$

Optimal portfolio choice

$w_0(1+i) + a(x-i)$

- Since the constraint will bind in the optimum, we can rewrite:

$$\max_a E[u(w_0(1+i) + a(x-i))]$$

$u(w(a))$

- The **first-order condition (FOC)** reads:

$$\frac{\partial \mathrm{E}u}{\partial a} = \mathrm{E}[\underbrace{u'}_{\partial u/\partial w}(\overbrace{w_0(1+i) + a^*(x-i)}^{w})(x-i)] \stackrel{!}{=} 0$$

- The **second-order condition (SOC)** reads:

$$\frac{\partial^2 \mathrm{E}u}{\partial a^2} = \mathrm{E}[\underbrace{u''(\cdot)}_{\ominus}\underbrace{(x-i)^2}_{\oplus}] \stackrel{!}{<} 0$$

- The SOC is satisfied for risk averse individuals ($u'' < 0$).

Important result

$E[c \cdot \tilde{x}] = c \cdot E(\tilde{x})$

- Hypothetical question: Can it be optimal that $a^* = 0$?

$$
\begin{aligned}
\frac{\partial \mathrm{E}u}{\partial a}\Big|_{a=0} &= \mathrm{E}[\overbrace{u'(w_0(1+i))}^{w}(\tilde{x}-i)] \\
&= u'(w_0(1+i))\mathrm{E}[x-i] \\
&= u'(w_0(1+i))(\mu - i)
\end{aligned}
$$

$E(\tilde{x}+c) = E(\tilde{x})+c$

$= E(\tilde{x}) - i$

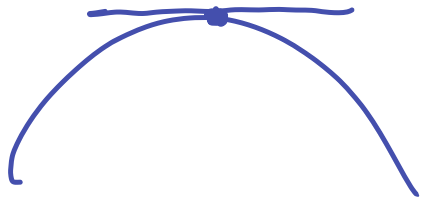
Can FOC $\overset{?}{=} 0$

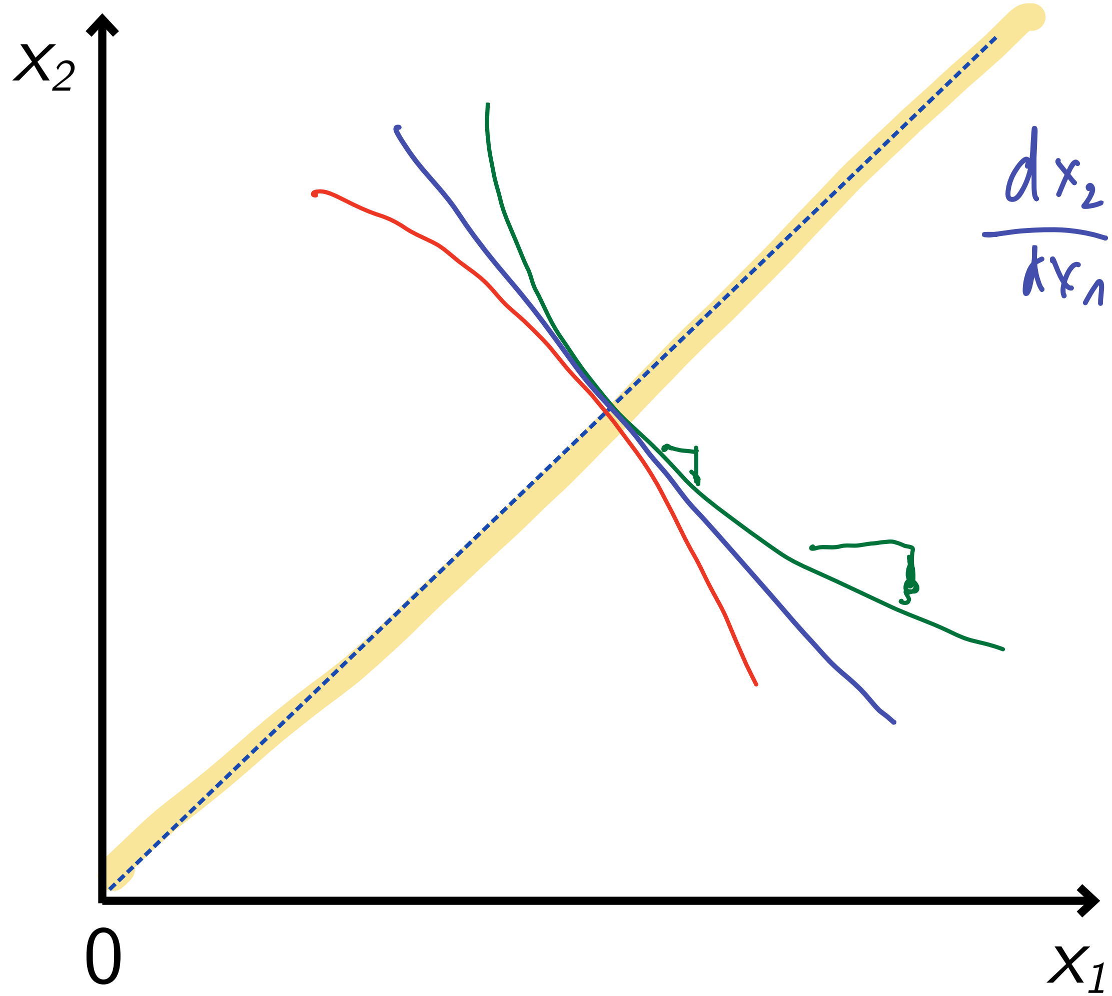
Only if $\mu = i$ ✓

Can FOC $\overset{?}{<} 0$

- Since $u' > 0$, it follows that
  - $a^* > 0 \iff \mu > i$, and
  - $a^* = 0 \iff \mu \leq i$ (If restricting a^* to non-negative values.)
- This is a **strong result:** Any risk-averse individual will (to some degree) partake in a risky project if the expected return from doing so is strictly larger than i.
- Intuition: At the certainty level, risk costs are second-order.

Graph: States of the World diagram





$$\left.\frac{dx_2}{dx_1}\right|_{x_1=x_2} = -\frac{(1-p)}{p}\cdot\cancel{\frac{u'(x_2)}{u'(x_2)}}$$

$= \text{const}$

A refresher: Implicit function theorem

- For the rest of this chapter, we will look at the **comparative static properties** of optimal portfolio choice
- For this, we need the **implicit function theorem (IFT).**

Lemma 4.1: Implicit function theorem

- Let $f(x, y)$ be a continuously differentiable function with $f(x, y) = 0$ and $\frac{\partial f}{\partial x}|_{x,y} \neq 0$.
- Then, the following equality will hold:

$$\frac{dx}{dy} = -\frac{\frac{\partial f(x,y)}{\partial y}}{\frac{\partial f(x,y)}{\partial x}}$$

Intuition of comparative static results

- However, the **proofs** (for a general utility function) are **technically very involved** and go beyond what is expected of you in this blocked course
- This is why, it will be **sufficient to discuss the intuition** of the comparative static results
- So **let's start discussing:** What do you believe to be true?
  - As risk aversion increases, a^* $\downarrow$
  - As the riskiness of the risky asset increases, a^* $\downarrow$
  - As initial income (w_0) increases, a^*
    - $\nearrow \uparrow$ for DARA
    - $\rightarrow -$ for CARA
    - $\searrow \downarrow$ for IARA
  - As the return of the safe asset (i) increases, a^*

$$\frac{\partial a^*}{\partial i} = \underset{-}{SE} + \underset{+}{IF}$$

- We will verify a few of these results in an exercise for $u(x) = \ln(x)$