Economic Foundations and Applications of Risk

Part B. Applications

Chapter 7: Allocation of Risk

Till Stowasser

LMU, 2023

Syllabus

- 7.1 Introduction
- 7.2 Efficient risk allocation
- 7.3 Arrow securities

7.1 Introduction

- In this chapter, we move from the individual perspective to the **social perspective on risk.**
- We will **characterize the efficient allocation of risk** between individuals if trade of state-dependent income were possible (7.2).
- Then we will introduce the concept of an **Arrow security**, which allows the aforementioned trade of risk (7.3).

7.2 Efficient risk allocation

Setup

- Let there be a simple exchange economy with **two individuals** (1 and 2) . . .
- . . . and two possible **states of the world** (a and b) that realize with probabilities p and $1-p$, respectively.
- Individual j's **initial endowment** in state i is $w_{0j}(i)$, . . .
- . . . and her **final wealth** (i.e. after trade) in state i is $w_{fj}(i)$.
- Individual j's **utility** is given by the at least twice differentiable function u_j, with $u_j' > 0 \geq u_j''$.

Social planner's problem

$$\max_{w_{f1}(a),w_{f1}(b),w_{f2}(a),w_{f2}(b)} pu_1(w_{f1}(a)) + (1-p)u_1(w_{f1}(b)) \quad \text{s.t.}$$

$$pu_2(w_{f2}(a))+(1-p)u_2(w_{f2}(b)) \geq pu_2(w_{02}(a))+(1-p)u_2(w_{02}(b)) \equiv \bar{u}$$

$$w_{f1}(a) + w_{f2}(a) \leq w_{01}(a) + w_{02}(a) \equiv w_0(a)$$

$$w_{f1}(b) + w_{f2}(b) \leq w_0(b)$$

$$\Longleftrightarrow$$

$$\max_{w_{f2}(a),w_{f2}(b)} pu_1(w_0(a) - w_{f2}(a)) + (1-p)u_1(w_0(b) - w_{f2}(b)) \quad \text{s.t.}$$

$$pu_2(w_{f2}(a)) + (1-p)u_2(w_{f2}(b)) \geq \bar{u}$$

j = (1,2) ← Individuals
i = (a,b) ← States of World

Optimal risk allocation

- Langrangian maximization yields the following **FOC:**

$$MRS_1 = \frac{pu_1'(a)}{(1-p)u_1'(b)} = \frac{pu_2'(a)}{(1-p)u_2'(b)} = MRS_2$$

- As always: The SOC holds because of risk aversion.
- This result also holds in the general case and is referred to as the **Borch condition:**
    - An allocation of risk is Pareto-efficient iff, in all possible states of the world, the marginal rate of substitution of income in state s and income in state t is the same for all individuals:

$$\forall i, j, s, t : \frac{p_s u_i'(w_{fi}(s))}{p_t u_i(w_{fi}(t))} = \frac{p_s u_j'(w_{fj}(s))}{p_t u_j'(w_{fj}(t))}$$

Some properties of optimal risk allocation

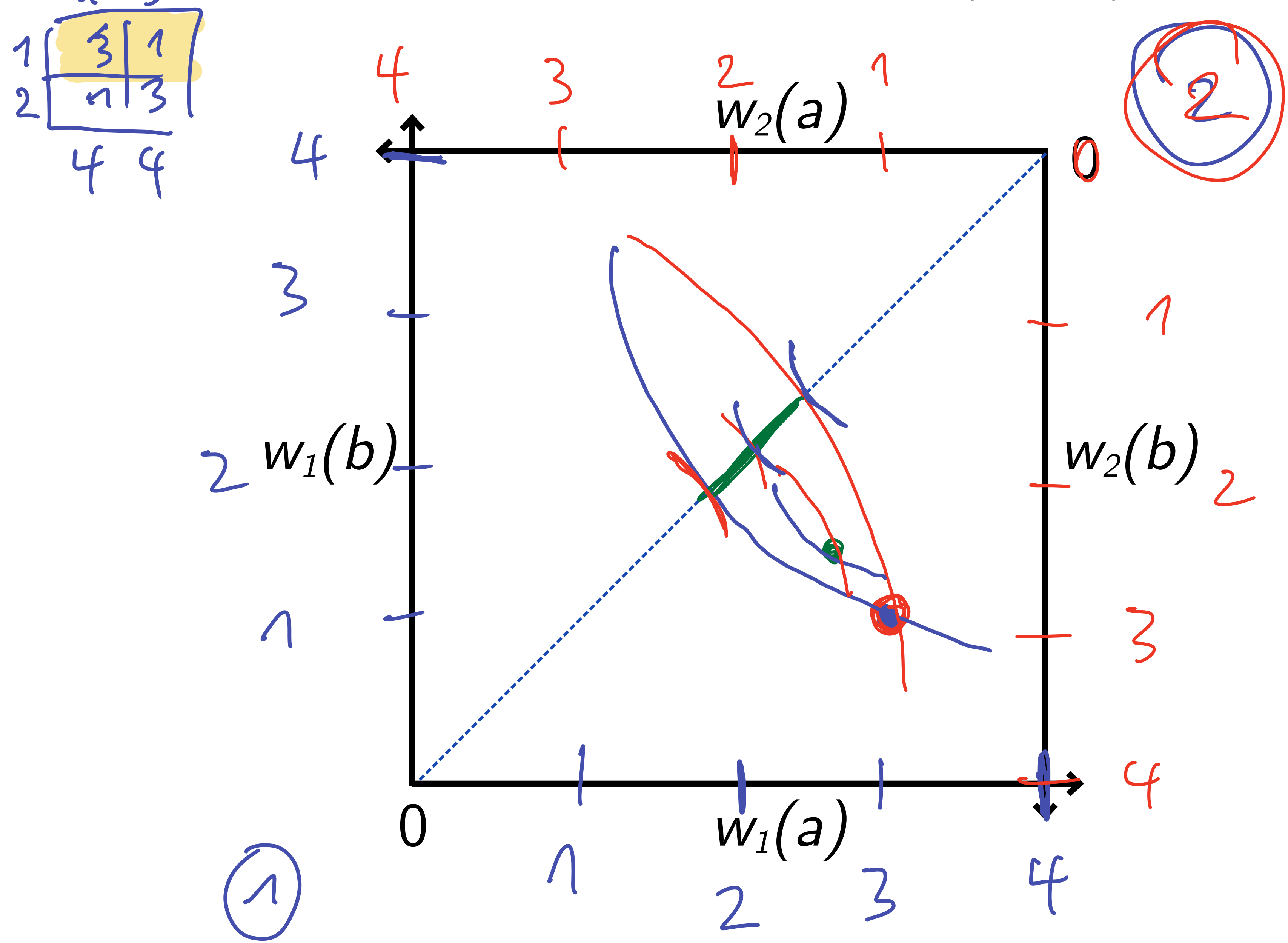
- The optimal allocation of risk between two individuals **depends** (among other things) **on** . . .
- . . . the **risk aversion** of individuals . . .
- . . . and the presence of **social risk.**
  - Social risk: For society as a whole, one state of the world is better than the other: $w_0(a) \neq w_0(b)$ (with $w_0(i) \equiv w_{01}(i) + w_{02}(i)$).
- Let us examine the following **four cases:**
  1. Both risk-averse & no social risk
  2. Both risk-averse & social risk
  3. One risk-neutral, one risk-averse & no social risk
  4. One risk-neutral, one risk-averse & social risk

Ad 1: Both risk-averse & no social risk

- With risk being allocated optimally, **no individual will bear any risk** if . . .
  - . . . both individuals are risk-averse, and
  - . . . there is no social risk.
- This is quite intuitive...

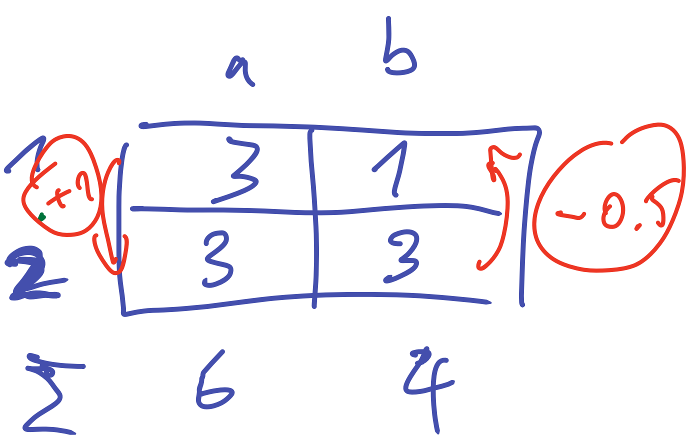
	a	b
1	3	1
2	1	3
Σ	4	4

Ad 1: Both risk-averse & no social risk (Graph)

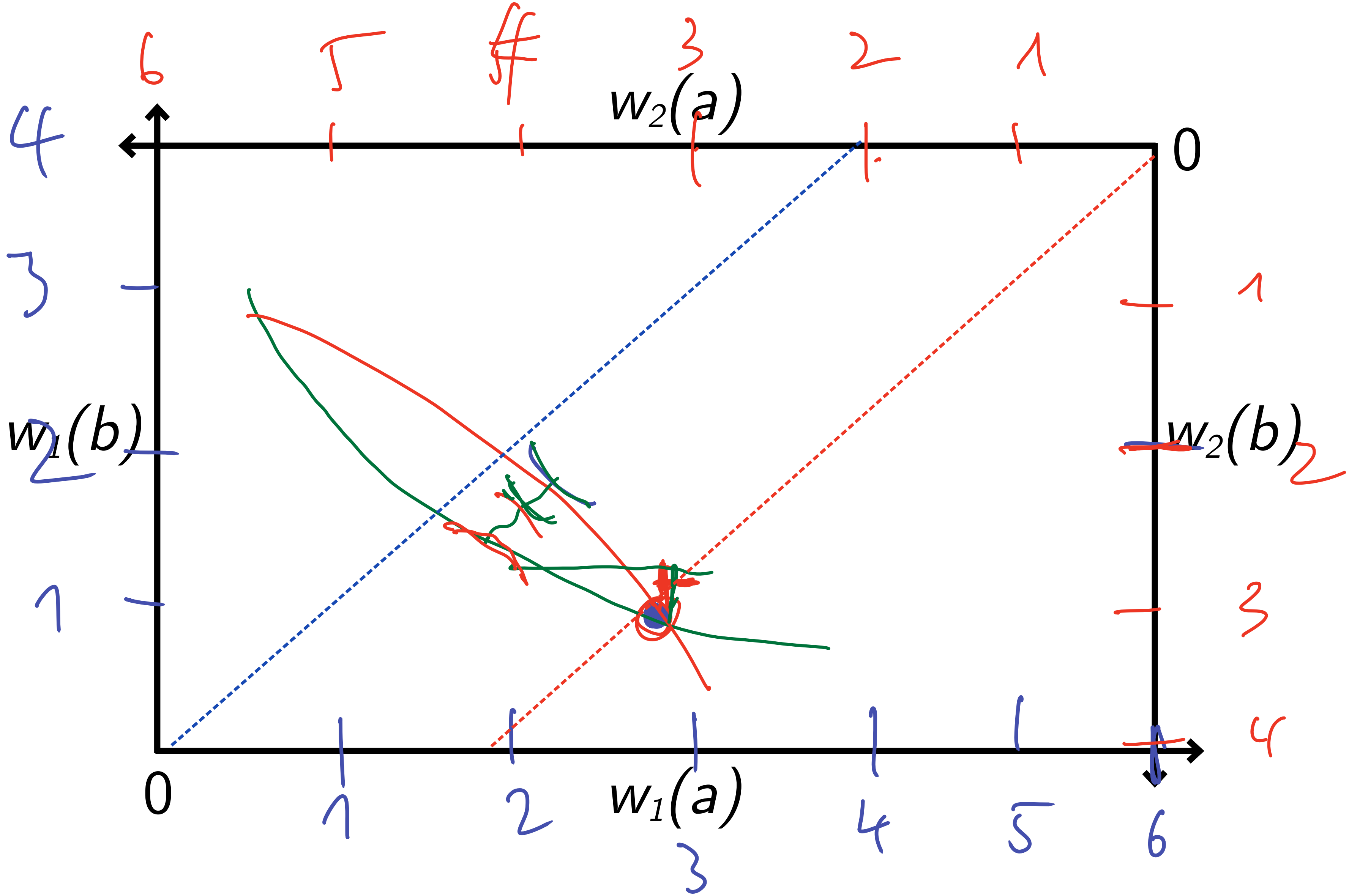


Ad 2: Both risk-averse & social risk

- With risk being allocated optimally, **both individuals will bear some risk** if . . .
  - . . . both individuals are risk-averse, and
  - . . . there is social risk.
- This also is quite intuitive...or is it???



Ad 2: Both risk-averse & social risk (Graph)



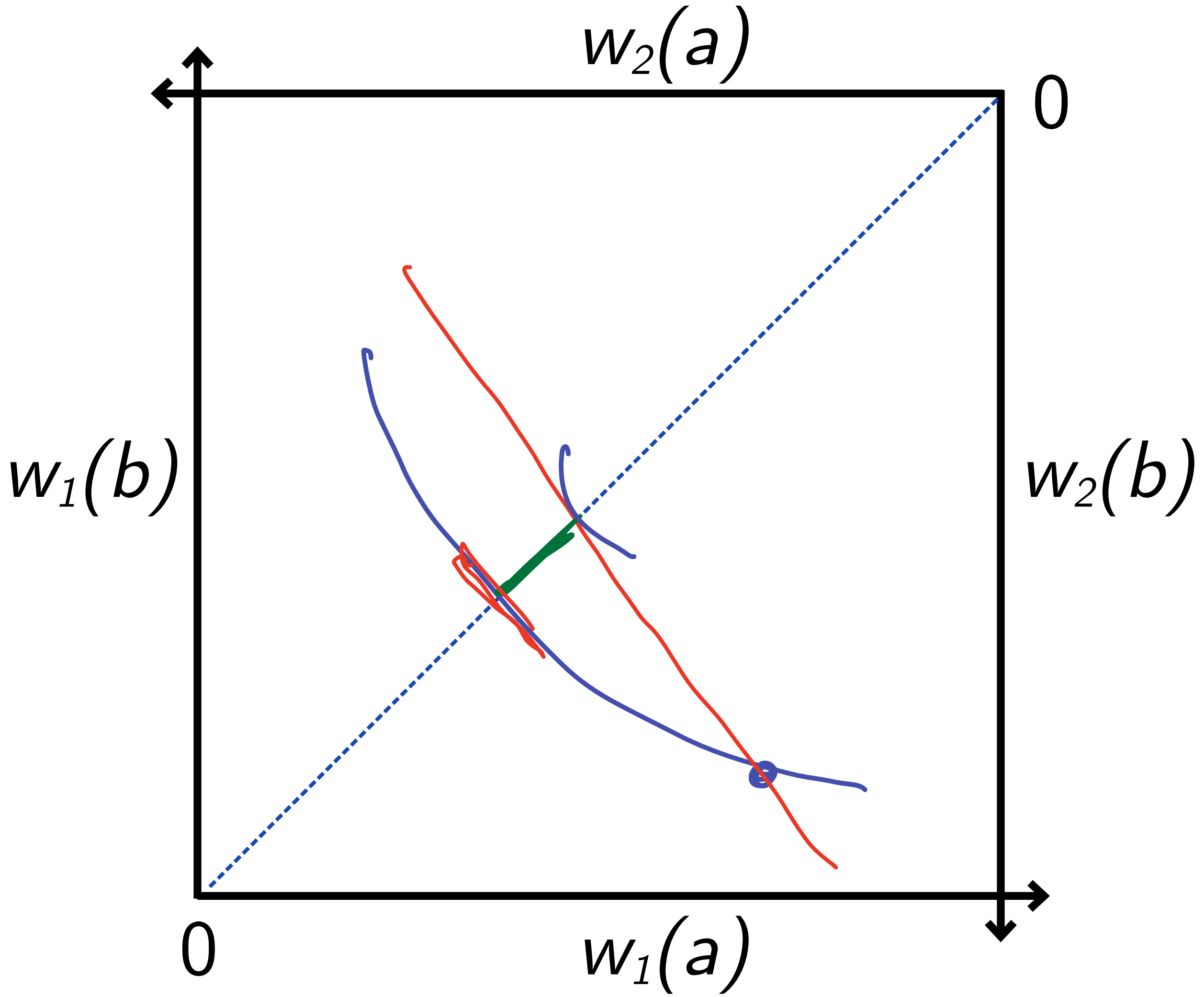
Ad 3 and 4: One risk-neutral, one risk-averse

- With risk being allocated optimally, **the risk-neutral individual will bear all the risk** if one individual is risk-averse and the other risk-neutral.
- This also is quite intuitive...
- **Proof:**
  - Suppose individual 2 is risk-neutral (implying that $u_2' = const$).
  - Then, the Borch condition simplifies to:

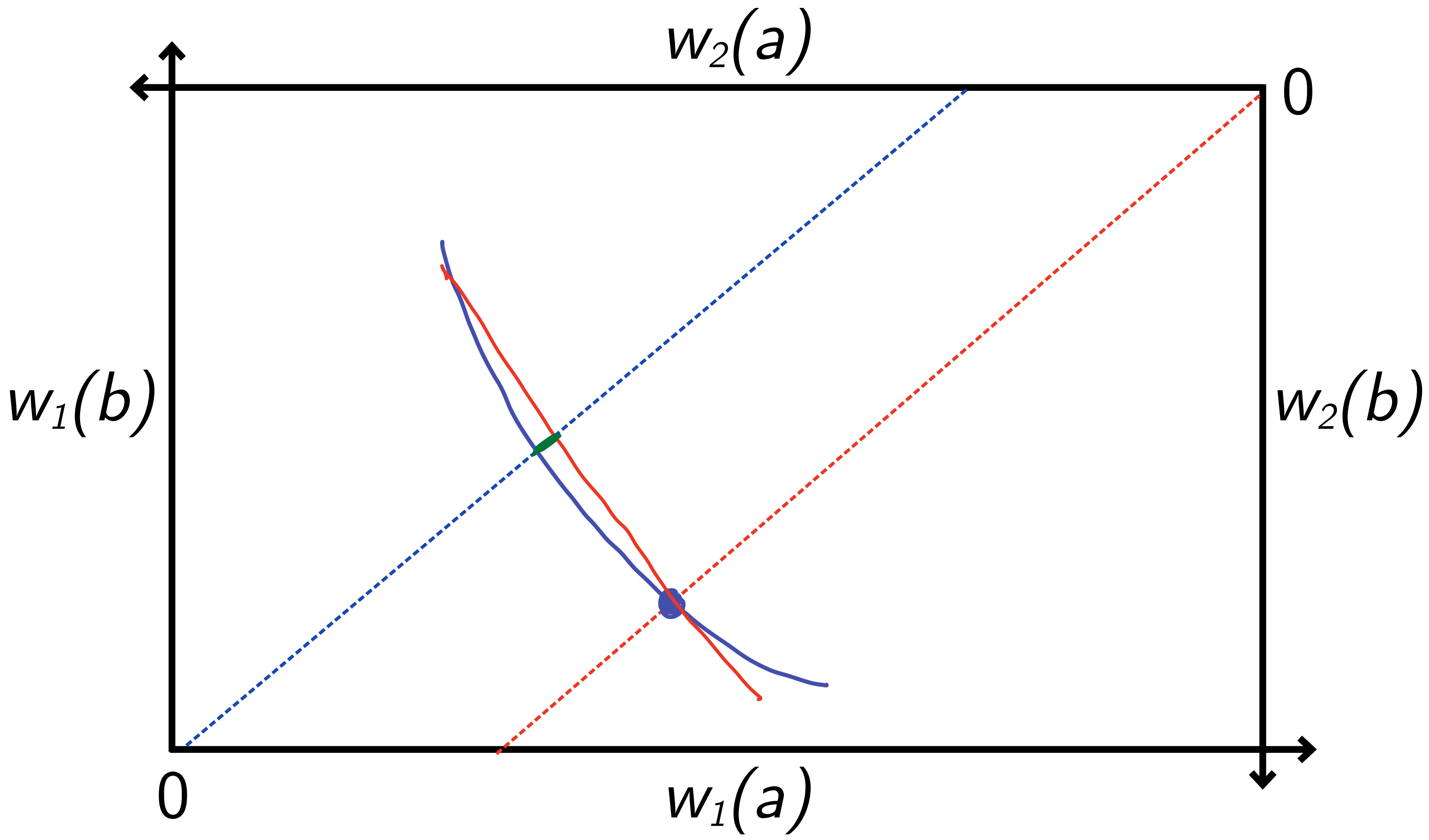
$$\frac{pu_1'(a)}{(1-p)u_1'(b)} = \frac{pu_2'(a)}{(1-p)u_2'(b)} = \frac{p}{1-p}.$$

  - Since individual 1 is risk-averse, $u_1'(a) = u_1'(b) \Rightarrow w_{f1}(a) = w_{f1}(b)$.
  - Hence, the risk-neutral agent will fully insure the risk-averse agent.

Ad 3: One risk-neutral, one risk-averse & no social risk (Graph)



Ad 4: One risk-neutral, one risk-averse & social risk (Graph)



7.3 Arrow securities

Motivation

- **Direct trade** of state-dependent income between individuals is **not realistic.**
- How can efficient risk-allocations be achieved?
- Arrow's idea: **Hypothetical asset** that is traded on the financial market which allows to move income along states: Arrow securities

Setup

- An **Arrow security** a_s for state of the world s is **defined as** a security that pays out unity in state of the world s and 0 in all other states of the world.
- Let q_s denote the **market price** of one Arrow security for state s.
  - If we assume competitive markets and there is no discounting of future payments, arbitrage will lead to $\sum_s q_s = 1$.
- Let $x_{fj}(s)$ denote individual j's **final wealth** in state of the world s, which results from her initial endowment as well as from trading with Arrow securities.

Optimization

- Individuals will maximize

$$\sum_s p_s u(x_{fj}(s)) \quad \text{s.t.}$$

$$\sum_s q_s(x_{fj}(s) - x_{0j}(s)) \leq 0$$

- First-order conditions:

$$\text{FOC for state s: } p_s u'(x_{fj}(s)) - \lambda_j q_s \stackrel{!}{=} 0$$

$$\text{FOC for state t: } p_t u'(x_{fj}(t)) - \lambda_j q_t \stackrel{!}{=} 0$$

- Dividing both FOCs yields:

$$\frac{p_s u'(x_{fj}(s))}{p_t u'(x_{fj}(t))} = \frac{q_s}{q_t}$$

- So: The market will lead to a result where MRS=price ratio.
- Hence, in equilibrium, marginal rates of substitution will be the same and the **Borch condition will hold.**
- Thus, with complete markets for Arrow securities, the **market will replicate the benevolent social planner's solution**.

Arrow securities in the real world

- In the real world there is no market designated for Arrow securities.
- However, **many real-world markets are equivalent** to a market for Arrow securities (e.g. insurance markets, equity markets, etc.)
- Hence, regular **financial markets will lead to an efficient risk allocation if there are enough linearly independent assets.**